Chapter 10
Review of Envelope Statistics Models for Quantitative Ultrasound Imaging and Tissue Characterization

François Destrempes and Guy Cloutier

Abstract The homodyned K-distribution and the K-distribution, viewed as a special case, as well as the Rayleigh and the Rice distributions, viewed as limiting cases, are discussed in the context of quantitative ultrasound (QUS) imaging. The Nakagami distribution is presented as an approximation of the homodyned K-distribution. The main assumptions made are: (1) the absence of log-compression or application of non-linear filtering on the echo envelope of the radiofrequency signal; (2) the randomness and independence of the diffuse scatterers. We explain why other available models are less amenable to a physical interpretation of their parameters. We also present the main methods for the estimation of the statistical parameters of these distributions. We explain why we advocate the methods based on the *X*-statistics for the Rice and the Nakagami distributions, and the K-distribution. The limitations of the proposed models are presented. Several new results are included in the discussion sections, with proofs in the appendix.

Keywords Quantitative ultrasound (QUS) · Ultrasound tissue characterization · Ultrasound imaging · Echo envelope · Homodyned K-distribution · K-distribution · Rice distribution · Rayleigh distribution · Nakagami distribution · Parameters estimation · Moments · Log-moments

F. Destrempes · G. Cloutier (✉)
Laboratory of Biorheology and Medical Ultrasonics, University of Montreal Hospital Research Center (CRCHUM), Montréal, Québec H2L-2W5, Canada
e-mail: guy.cloutier@umontreal.ca

F. Destrempes
e-mail: francois.destrempes@crchum.qc.ca

G. Cloutier
The Department of Radiology, Radio-Oncology and Nuclear Medicine, University of Montreal, Montréal, Québec H3T-1J4, Canada

G. Cloutier
The Institute of Biomedical Engineering, University of Montreal, Montréal, Québec H3T-1J4, Canada

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10.1 Introduction

The statistical distributions presented here appeared in the context of various applications in the past 130 years or so. The Rayleigh distribution was introduced in Rayleigh (1880) in the context of sound propagation. The Rice distribution appeared in Nakagami (1940), and in Rice (1945) as a model of wave propagation. The K-distribution was first introduced in Lord (1954) in the context of random walks and then in the context of sea echo (Jakeman and Pusey 1976). The homodyned K-distribution was introduced and studied in Jakeman (1980) and Jakeman and Tough (1987) as a model of weak scattering. The Nakagami distribution was defined in Nakagami (1943) in the field of wave propagation.

In ultrasound imaging, the Rayleigh distribution appeared as a model of the gray level (also called amplitude) in an unfiltered B-mode image, viewed as the envelope of the radiofrequency (RF) image, in the case of a high density of random scatterers with no coherent signal component (Burckhardt 1978; Wagner et al. 1983). The Rice distribution corresponds to a high density of random scatterers (the diffuse signal component), but combined with the presence of a coherent signal component (Insana et al. 1986). Non-Rayleigh distributions were considered in ultrasound imaging as early as the pioneer article of Burckhardt (1978). The K-distribution corresponds to a variable (effective) density α of random scatterers, with no coherent signal component and was introduced in ultrasound imaging by Shankar et al. (1993), and Molthen et al. (1993), Narayanan et al. (1994), Shankar (1995) and Molthen et al. (1995). The homodyned K-distribution corresponds to the general case of a variable effective density of random scatterers with or without a coherent signal component (Dutt and Greenleaf 1994). A simpler model consists in modeling the gray level of the speckle pattern in a B-mode image by a Nakagami distribution (Shankar 2000; Dumane and Shankar 2001). In the context of QUS, the estimated parameters of the statistical distribution of the echo envelope play the role of quantitative measures that give information about the underlying tissues of interest.

The Nakagami distribution has been the most frequently adopted model in the context of tissue characterization, probably due to its simplicity. As the pioneer work, let us mention Shankar et al. (2001) in the context of breast tumor classification. The Nakagami model was then systematically used in various medical ultrasound imaging fields, including: ophthalmology (Tsui et al. 2007); vascular flow applications (Huang et al. 2007; Huang and Wang 2007; Tsui et al. 2008a, 2009a, 2009b); and breast cancer (Tsui et al. 2008b, 2010a, 2010b, 2010c). The K-distribution was used in the context of breast cancer classification in the pioneer work of Shankar et al. (1993). More recently, the homodyned K-distribution was used for cardiac tissue characterization (Hao et al. 2002) and cancerous lesion classification (Oelze and O'Brien 2007; Hruska et al. 2009; Mamou et al. 2010, 2011), and a model of mixtures of Rayleigh distributions was adopted for liver fibrosis quantification (Yamaguchi et al. 2011).

Whereas at the time of Shankar et al. (1993) and Dutt and Greenleaf (1994), the estimation of the (homodyned) K-distribution was a problem, since then, several estimation methods have been published. In particular, the K-distribution can be estimated using the simple and reliable X-statistics (Blacknell and Tough 2001) (defined as the log-moment $E[I \log I]/E[I] - E[\log I]$, where I denotes the square of the echo envelope amplitude) and a method by Hruska and Oelze (2009) was proposed for the estimation of the homodyned K-distribution. Thus, the use of the Nakagami model does not seem justified anymore in the context of QUS, since it reduces the information carried by the homodyned K-distribution model.

10.2 Chapter Content

The remaining part of this chapter is organized as follows. Section 10.3 presents in details the various models mentioned in the introduction, as well as other available models. A physical underlying model is also presented to help with the interpretation of the various parameters. In Sect. 10.4, the most frequently used estimation methods for the main distributions are presented. Finally, Sect. 10.5 presents the limitations of the main models and hints to future work on that matter.

In Sect. 10.4.1, various estimation methods are explained: (1) the Maximum Likelihood Estimator (MLE) and the Maximum A Posteriori (MAP); (2) moments based methods; and (3) log-moments based methods. Then, in Sects. 10.4.2–10.4.6, we kept the same structure in the presentation of the estimation methods for each of the five distributions presented in Sects. 10.3.1 and 10.3.3 whenever possible (according to the literature).

In Sects. 10.3 and 10.4, various remarks and additional results are mentioned in the subsections entitled “discussion“. As far as we know, most of these results are new (Theorems 8-12, 18-23, 25-26, Corollary 2, Lemmas 4 and 5), except probably Lemmas 1-3, although we did not find references. Theorem 28 was used explicitly in Destrempes et al. (2009, Table 1), but without proof. The proofs of the new results are postponed to the appendix. The purpose of these new results is to deepen the understanding of the published methods mentioned in this chapter.

Table 10.1 Main distributions discussed in this chapter and the relations among them

Distribution	Notation	Relation
Gamma	$\mathcal{G}(w\|\alpha, 1)$	
Rice	$P_{\mathrm{Ri}}(A\|\varepsilon, \sigma^2)$	
Rayleigh	$P_{\mathrm{Ra}}(A\|\sigma^2)$	$= P_{\mathrm{Ri}}(A\|0, \sigma^2)$
Homodyned K	$P_{\mathrm{HK}}(A\|\varepsilon, \sigma^2, \alpha)$	$= \int_0^\infty P_{\mathrm{Ri}}(A\|\varepsilon, \sigma^2 w)\mathcal{G}(w\|\alpha, 1)\, dw$
K	$P_{\mathrm{K}}(A\|\sigma^2, \alpha)$	$= \int_0^\infty P_{\mathrm{Ra}}(A\|\sigma^2 w)\mathcal{G}(w\|\alpha, 1)\, dw$
		$= P_{\mathrm{HK}}(A\|0, \sigma^2, \alpha)$
Rice	$P_{\mathrm{Ri}}(A\|\varepsilon, \sigma^2)$	$= \lim_{\alpha\to\infty} P_{\mathrm{HK}}(A\|\varepsilon, \sigma^2/\alpha, \alpha)$
Rayleigh	$P_{\mathrm{Ra}}(A\|\sigma^2)$	$= \lim_{\alpha\to\infty} P_{\mathrm{K}}(A\|\sigma^2/\alpha, \alpha)$
Nakagami	N $(A\|m, \Omega)$	Approximation of the homodyned K-distribution

10.3 Statistical Models

We first present in Sect. 10.3.1 various models for the first-order statistics of the amplitude of the echo envelope. The most general of these models is the homodyned K-distribution that depends on three parameters $\varepsilon \geq 0$, $\sigma^2 > 0$ and $\alpha > 0$. The Rice distribution with parameter $\sigma^2 = \overline{a^2}/2$ is viewed as the limiting case of the homodyned K-distribution with parameter σ^2 of the form $\overline{a^2}/(2\alpha)$ when $\alpha \to \infty$. The K-distribution is a special case of the homodyned K-distribution with $\varepsilon = 0$, whereas the Rayleigh distribution is a special case of the Rice distribution with the same constraint. The parameter α is related to the homogeneity of the diffuse scattering medium and the density of the scatterers. It is called the *scatterer clustering parameter* (Dutt and Greenleaf 1994). In the context of the K-distribution, it is also called the *effective number of scatterers* (Narayanan et al. 1994). The parameter ε is called the *coherent component* and is related to the strength of the specular reflection or the periodic organization of the scatterers. The diffuse signal power can be viewed as $2\sigma^2\alpha$ for the homodyned K-distribution (and K-distribution), whereas ε^2 can be viewed as the coherent signal power. In Sect. 10.3.2, an underlying physical model is presented. In Sect. 10.3.3, the Nakagami distribution is described as an approximation of the homodyned K-distribution. Finally, in Sect. 10.3.4, the relevance of these distributions compared to other available models is discussed (see Destrempes and Cloutier (2010) for further reading on that topic). The reader may refer to Table 10.1 for a summarize of the main distributions discussed here and the relations among them.

10.3.1 The Homodyned K-Distribution and Related Distributions

10.3.1.1 The Rayleigh Distribution

The (2-dimensional) Rayleigh distribution (Rayleigh 1880) is defined by

$$P_{\mathrm{Ra}}(A \mid \sigma^2) = \frac{A}{\sigma^2} \exp\left(-\frac{A^2}{2\sigma^2}\right), \tag{10.1}$$

where A represents the amplitude of the signal. In Jakeman and Tough (1987), the distribution is expressed, in the context of n-dimensional random walks, in terms of the variable $\overline{a^2} = n\sigma^2$. The case $n = 2$ corresponds to Eq. (10.1). Equivalently, the intensity I, i.e. the square of the amplitude A, is distributed according to an exponential distribution.

10.3.1.2 The Rice Distribution

The (2-dimensional) Rice distribution is expressed as

$$P_{\mathrm{Ri}}(A \mid \varepsilon, \sigma^2) = \frac{A}{\sigma^2} I_0\Big(\frac{\varepsilon}{\sigma^2} A\Big) \exp\Big(-\frac{(\varepsilon^2 + A^2)}{2\sigma^2}\Big), \tag{10.2}$$

where $\sigma > 0$ and $\varepsilon \geq 0$ are real numbers and I_0 denotes the modified Bessel function of the first kind of order 0 (the intensity I should not be confused with the Bessel function I_0). See Jakeman and Tough (1987, Eq. 2.16) for a generalization in dimension $n \geq 2$. The special case where $\varepsilon \to 0$ yields the Rayleigh distribution. The case $n = 2$ corresponds to Nakagami (1940) and Rice (1945). In Nakagami (1960, p. 4, Eq. 5), the Rice distribution is called the "n-distribution" (Nakagami, 1940).

10.3.1.3 The K-Distribution

The (2-dimensional) K-distribution (Lord 1954; Jakeman and Pusey 1976) is defined by

$$P_{\mathrm{K}}(A \mid \sigma^2, \alpha) = \frac{4A^{\alpha}}{(2\sigma^2)^{(\alpha+1)/2}\Gamma(\alpha)} K_{\alpha-1}\Big(\sqrt{\frac{2}{\sigma^2}} A\Big), \tag{10.3}$$

where $\alpha > 0$, $\sigma^2 > 0$, Γ is the Euler gamma function, and K_p denotes the modified Bessel function of the second kind of order p. In Jakeman and Tough (1987, Eq. 2.11), the distribution is expressed in terms of the parameters α and $b = \sqrt{\frac{2}{\sigma^2}}$. In view of the compound representation presented below, we find the proposed parametrization more convenient.

Theorem 1 (Jakeman and Tough 1987). *The compound representation of the K-distribution is*

$$P_{\mathrm{K}}(A \mid \sigma^2, \alpha) = \int_0^{\infty} P_{\mathrm{Ra}}(A \mid \sigma^2 w) \mathcal{G}(w \mid \alpha, 1)\, dw, \tag{10.4}$$

where P_{Ra} denotes the Rayleigh distribution, and $\mathcal{G}(w \mid \alpha, 1)$ is the gamma distribution $w^{\alpha-1} \exp(-w)/\Gamma(\alpha)$ of mean and variance equal to α.

The compound representation is useful to simulate the K-distribution, and in the evaluation of its value. A K-distribution with parameters $\sigma^2 = \overline{a^2}/(2\alpha)$ and α yields the Rayleigh distribution with parameter "σ^2" $= \overline{a^2}/2$, as $\alpha \to \infty$. See Jakeman and Tough (1987, Eq. 2.12). Thus, the parameter "σ^2" of the limiting Rayleigh distribution is $\overline{a^2}/2$, and should not be confused with the parameter σ^2 of the K-distribution. The relation between these two quantities is "σ^2" $= \overline{a^2}/2 = \sigma^2 \alpha$.

10.3.1.4 The Homodyned K-Distribution

The (2-dimensional) homodyned K-distribution (Jakeman 1980; Jakeman and Tough 1987) is defined by

$$P_{\mathrm{HK}}(A \mid \varepsilon, \sigma^2, \alpha) = A \int_0^\infty u J_0(u\varepsilon) J_0(uA) \left(1 + \frac{u^2 \sigma^2}{2}\right)^{-\alpha} du \tag{10.5}$$

where $\sigma^2 > 0$, $\alpha > 0$, $\varepsilon \geq 0$, and J_0 denotes the Bessel function of the first kind of order 0. In Jakeman and Tough (1987, Eq. 4.13), the homodyned K-distribution is expressed in terms of the parameters α, $\overline{a^2} = n\sigma^2\alpha$, and $a_0 = \varepsilon$, in the context of n-dimensional random walks.

Theorem 2 (Jakeman and Tough 1987). *The compound representation of the homodyned K-distribution is*

$$P_{\mathrm{HK}}(A \mid \varepsilon, \sigma^2, \alpha) = \int_0^\infty P_{\mathrm{Ri}}(A \mid \varepsilon, \sigma^2 w) \mathcal{G}(w \mid \alpha, 1) d\, w, \tag{10.6}$$

where P_{Ri} denotes the Rice distribution and $\mathcal{G}(w \mid \alpha, 1)$ is the gamma distribution with mean and variance equal to α.

The case $\varepsilon \to 0$ yields the K-distribution (with parameters σ^2, α). In particular, the compound representation of the homodyned K-distribution is consistent with Eq. (10.4), upon taking $\varepsilon \to 0$. A homodyned K-distribution with parameters ε, $\sigma^2 = \overline{a^2}/(2\alpha)$ and α yields the Rice distribution with parameters ε and "σ^2" $= \overline{a^2}/2$, as $\alpha \to \infty$. Thus, if in addition, $\varepsilon \to 0$, then one obtains the Rayleigh distribution with parameter "σ^2" $= \overline{a^2}/2$, as $\alpha \to \infty$. Figure 10.1 illustrates four representative examples of the homodyned K-distribution (including two examples of the K-distribution, as a special case).

Two functions of the three parameters of the homodyned K-distribution are invariant under scaling of the intensity (Dutt and Greenleaf 1994): (1) the parameter α; (2) the structure parameter $\kappa = \varepsilon^2/(2\sigma^2\alpha)$, i.e. the ratio of the coherent signal power ε^2 with the diffuse signal power $\overline{a^2} = 2\sigma^2\alpha$. Other parameters of the homodyned K-distribution were considered in the literature: the coherent to diffuse signal ratio $k = \sqrt{2\kappa} = \varepsilon/(\sigma\sqrt{\alpha})$ (Dutt and Greenleaf 1994; Dutt 1995; Hruska and Oelze 2009), and the parameter β equal to $1/\alpha$ (Dutt and Greenleaf 1994; Dutt 1995).

Considering $\sigma^2 = \overline{a^2}/(2\alpha)$ and letting α tend to infinity, one obtains a Rice distribution for which the diffuse signal power is also $\overline{a^2}$ and the structure parameter κ is also equal to $\varepsilon^2/\overline{a^2}$.

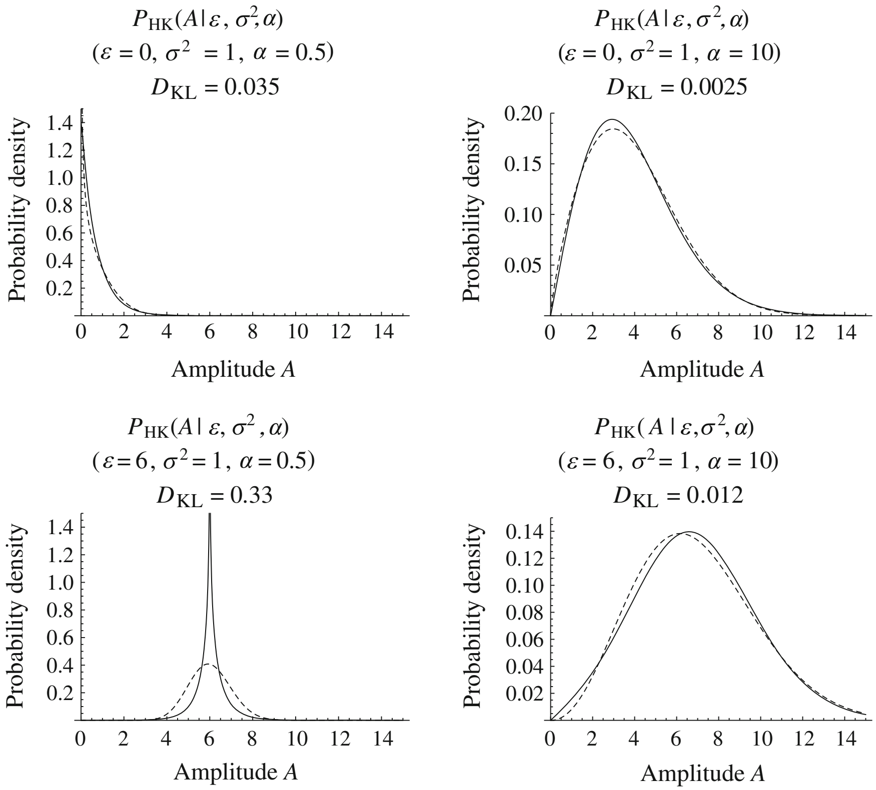

Fig. 10.1 Typical examples of the homodyned K-distribution. *Top row* K-distribution ($\varepsilon = 0$). *Bottom row* $\varepsilon > 0$. The *dashed curves* represent the approximating Nakagami distributions. The Kullback-Leibler distance values between the two distributions were: *top left* 0.035; *top right* 0.0025; *bottom left* 0.33 (with a coherent to diffuse signal ratio $k = \varepsilon/(\sigma\sqrt{\alpha})$ equal to 8.5); *bottom right* 0.012 (with $k \approx 1.9$)

10.3.2 Interpretation of the Distributions in the Context of Ultrasound Imaging

In Shankar et al. (1993), Molthen et al. (1993), Narayanan et al. (1994), Shankar (1995) and Molthen et al. (1995), one considers N_s scatterers lying in an ambient scattering medium within the resolution cell. Each scatterer corresponds to a phasor $a_j e^{i\phi_j}$ with two elements: (1) an amplitude a_j depending on the scattering properties (i.e., the scattering cross section) and the position of the scatterer with respect to the ultrasound beam, the instrumentation and the attenuation; (2) a phase ϕ_j that depends on the scatterer's position. One then postulates (Narayanan et al. 1994) a K-distribution with parameters σ^2 and α_s for each amplitude and considers uniformly distributed independent phases for each scatterer. The choice of the

K-distribution was motivated in Narayanan et al. (1994) by its good modeling properties of the first-order statistics of the echo envelope in the case where the Rayleigh distribution model (corresponding to infinitely many scatterers of identical cross-sections) breaks down, as explained in the next paragraph. Assuming weak scattering, the resulting complex signal is expressed as

$$\mathbf{A} = \sum_{j=1}^{N_s} a_j e^{i\phi_j}. \tag{10.7}$$

Then, its amplitude A is viewed as the norm of the complex signal.

Note that Eq. (10.7) can be viewed as a random walk in the Euclidean plane (corresponding to $n = 2$), since the complex number $e^{i\phi_j}$ corresponds to the vector $(\cos(\phi_j), \sin(\phi_j))$. Thus, the amplitude $A = \mathbf{A}$ corresponding to Eq. (10.7) follows a K-distribution with parameter $\alpha = \alpha_s N_s$. In that context, α is called the *effective number of scatterers*, because the number of scatterers N_s is multiplied by the parameter α_s. For instance, if N_s is large, but α_s is so small that $\alpha_s N_s \ll 10$, then the resulting distribution is a K-distribution rather than a Rayleigh distribution. For the same reason, even if N_s is small, but α_s is so large that $\alpha_s N_s \geq 10$, then one obtains a Rayleigh distribution. The parameter α_s is a parameter describing the lack of uniformity of the scattering cross sections in the range cell (c.f. Narayanan et al. 1994, Eq. (6)). A small value of α_s corresponds to a great variability, whereas a large value of α_s corresponds to a small variability. Thus, the parameter α_s is viewed as a measure of homogeneity of the scattering medium. The choice of the K-distribution is also consistent with the observation that the higher order moments of the intensity of the echo envelope may be larger than the ones predicted by the Rayleigh distribution model in the case of pathological tissues (Shankar et al. 1993). For instance, under the Rayleigh model, one would have $E[I^2]/E[I]^2 = 2$, whereas the K-distribution model yields $E[I^2]/E[I]^2 = 2(1 + 1/\alpha)$, which corresponds to observed values upon taking α sufficiently small. So, the statistics of the echo envelope depart from the Rayleigh model if the number of scatterers is small and α_s is not too large, or if the cross-sections are heterogeneous and N_s is not too large.

Adding a coherent component $\boldsymbol{\varepsilon}$, with constant amplitude ε and either a constant phase or a uniformly distributed phase, then yields a homodyned K-distribution with parameters ε, σ^2 and $\alpha = \alpha_s N_s$, for the amplitude $A = \|\boldsymbol{\varepsilon} + \mathbf{A}\|$. Since a coherent component may arise when the scatterers are organized periodically, the parameter α does not quite represent the effective number of scatterers in that context, but it may still be viewed as a scatterer clustering parameter. The coherent component may also be caused by specular reflection.

10.3.3 The Nakagami Distribution as an Approximation

The Nakagami distribution (Nakagami 1943, 1960) is defined by

$$\mathrm{N}\ (A\,|\,m,\Omega) = \frac{2m^m}{\Gamma(m)\Omega^m} A^{2m-1} e^{-mA^2/\Omega}, \tag{10.8}$$

where Γ is the Euler gamma function. The real numbers $m > 0$ and $\Omega > 0$ are called the shape parameter and the scaling parameter, respectively. Equivalently, the intensity $I = A^2$ follows a gamma distribution, with shape parameter m and scale parameter Ω/m.

The shape parameter m can be interpreted as the square of the intensity signal-to-noise ratio (SNR), i.e. $m = \frac{E[I]^2}{Var(I)}$ and Ω represents the mean intensity $E[I]$ (i.e., the total signal power). The intensity SNR should not be confused with the amplitude SNR. For instance, when $m = 1$, one retrieves the Rayleigh distribution. This observation can be found in Nakagami (1960, p. 17, Eqs. 50 and 51). In that case, the intensity SNR is equal to 1, whereas the amplitude SNR is about 1.91.

The Nakagami distribution can be viewed as an approximation of the homodyned K-distribution. First of all, we have the limiting case where $\alpha \to \infty$, which yields the Rice distribution and corresponds to the case $m \geq 1$.

Theorem 3 (Destrempes and Cloutier 2010). *Let* $m = \frac{(\varepsilon^2+2\sigma^2)^2}{4\sigma^2(\varepsilon^2+\sigma^2)}$ *and* $\Omega = \varepsilon^2 + 2\sigma^2$. *Then,*

$$\mathcal{D}_{KL}\Big(P_{\mathrm{Ri}}(\varepsilon,\sigma^2),\ \mathrm{N}\ (m,\Omega)\Big) \leq 0.02, \tag{10.9}$$

where $\mathcal{D}_{KL}$ *denotes the Kullback-Leibler distance* (Kullback and Leibler 1951) *between two distributions.*

Recall that the Kullback-Leibler distance (also called Kullback-Leibler divergence because it is non-symmetric) is a measure of the difference between two probability distribution functions (PDF) $f(x)$ and $g(x)$ in the continuous random variable x and is defined by $\int f(x) \log \frac{f(x)}{g(x)}\, dx$. It has the properties: (1) $\mathcal{D}_{\mathrm{KL}}(f,g) \geq 0$ for any PDFs f and g; and (2) $\mathcal{D}_{\mathrm{KL}}(f,g) = 0$ if and only if $f \equiv g$. However, it is a non-symmetric measure (i.e., $\mathcal{D}_{\mathrm{KL}}(f,g)$ is not necessarily equal to $\mathcal{D}_{\mathrm{KL}}(g,f)$). The choice of m and Ω in Theorem 3 is consistent with the identities $SNR^2 = \frac{(\varepsilon^2+2\sigma^2)^2}{4\sigma^2(\varepsilon^2+\sigma^2)}$ and $E[I] = \varepsilon^2 + 2\sigma^2$ valid for the Rice distribution. See also Nakagami (1960, p. 18, Eq. (55)).

We also have the following approximation result in the case of the K-distribution ($\varepsilon \to 0$), which corresponds to the case $m < 1$.

Theorem 4 (Destrempes and Cloutier 2010). *Let* $m = \frac{\alpha}{(\alpha+1)}$ *and* $\Omega = 2\sigma^2\alpha$. *Then,*

$$\mathcal{D}_{\mathrm{KL}}\Big(P_{\mathrm{K}}(\sigma^2,\alpha),\ \mathrm{N}\ (m,\Omega)\Big)\leq 0.0325. \tag{10.10}$$

For Theorem 4, the choice of m and Ω is consistent with the identities $E[I\log I]/E[I]-E[\log I]=1/m$ and $E[I\log I]/E[I]-E[\log I]=1+1/\alpha$ valid for the Nakagami distribution and the K-distribution, respectively (see Sects. 10.4.4.5 and 10.4.6.3).

10.3.4 Discussion

In this section, a new result on the Nakagami distribution as an approximation of the homodyned K-distribution is introduced (in greater generality than Theorems 3 and 4). We then discuss the consistency of the distributions presented in Sect. 10.3.1 in the limit case of a vanishing diffuse signal power, and we explain why other models available in the literature fail to have this feature (Destrempes and Cloutier 2010). Finally, as a new result, that property is shown to hold for the Nakagami distribution.

Considering the general case of a homodyned K-distribution with parameters ε, σ^2, and α, the $M^{(1)}$-statistics $E[A]/\sqrt{E[I]}$ is expressed explicitly in Theorem 24 as a function $M^{(1)}_{\mathrm{HK}}(\gamma,\alpha)$, where $\gamma=\kappa\alpha$. Using the identity $M^{(1)}_{\mathrm{Na}}(m)=\frac{\Gamma(1/2+m)}{\sqrt{m}\Gamma(m)}$ of Theorem 30, one then solves the equation $M^{(1)}_{\mathrm{Na}}(m)=M^{(1)}_{\mathrm{HK}}(\gamma,\alpha)$ in the variable m, using a binary search algorithm based on Theorem 31. This yields a function $m=m(\gamma,\alpha)$. Moreover, considering $E[I]$, one is led to the identity $\Omega=\varepsilon^2+2\sigma^2\alpha=\mu$ (the average value of the intensity). So, one is interested in the Kullback-Leibler distance

$$\mathcal{D}_{\mathrm{KL}}\Big(P_{\mathrm{HK}}(\varepsilon=\sqrt{\frac{\mu\gamma}{(\gamma+\alpha)}},\sigma^2=\frac{\mu}{2(\gamma+\alpha)},\alpha),\ \mathrm{N}\ (m=m(\gamma,\alpha),\Omega=\mu)\Big), \tag{10.11}$$

as a function of γ, α, and μ. It can be seen that this function is independent of the scaling factor μ (this is actually a general property of the Kullback-Leibler distance).

We computed Eq. (10.11) for $k=\varepsilon/(\sigma\sqrt{\alpha})=0.0,0.1,\ldots,2.0$, $\alpha=1,2,\ldots,20$, taking $\sigma^2=1/\alpha$. For each value of k and α, a sample set of size $N=1{,}000$ was simulated according to the corresponding homodyned K-distribution. The Kullback-Leibler distance was then estimated as the average over the simulated set of $\log\Big(P_{\mathrm{HK}}(A_i\,|\,\varepsilon=k,\sigma^2=1/\alpha,\alpha)/\ \mathrm{N}\ (A_i\,|\,m=m(\gamma,\alpha),\Omega=k^2+2)\Big)$. The maximal value was 0.072 (this result could be slightly improved upon considering the X-statistics instead of the $M^{(1)}$-statistics). So, the Nakagami pdf is a satisfying approximation in that range of the parameters k and α. See Fig. 10.1 for examples of approximating Nakagami distributions. In that figure, we included an example of a value of k much larger than 2; in that case, the KL distance is quite large (0.33).

Theorem 2 states that the homodyned K-distribution corresponds to a model in which the diffuse signal power $2\sigma^2$ of a Rice distribution is modulated by a gamma distribution, but not its coherent signal component ε. As mentioned in Destrempes and Cloutier (2010), there are several more models for the first-order statistics of the echo envelope. One modeling possibility introduced in Barakat (1986) and further developed in Jakeman and Tough (1987) is equivalent to modulate both the coherent signal component ε and the diffuse signal power $2\sigma^2$ of the Rice distribution by a gamma distribution. This gives rise to the generalized K-distribution. Note that this distribution has not been used in ultrasound imaging as of now. However, in Eltoft (2005), the Rician inverse Gaussian distribution (RiIG) is introduced, and it corresponds to a model in which both the coherent signal component ε and the diffuse signal power $2\sigma^2$ of a Rice distribution are modulated by an inverse Gaussian (IG) distribution, instead of a gamma distribution. Thus, this model is related to the generalized K-distribution, as further discussed in Destrempes and Cloutier (2010).

Three other distributions were introduced in the context of ultrasound imaging. The first one is called the generalized Nakagami distribution (Shankar 2000), and is obtained from the Nakagami distribution by a change of variable of the form $y = A^{1/s}$, where s is a shape adjustment parameter and A is the amplitude of the signal. This distribution was also proposed in Raju and Srinivasan (2002) (in the equivalent form of a generalized gamma distribution). The second other distribution is called the Nakagami-gamma distribution (Shankar 2003). That distribution can be viewed as a model in which the Rice distribution is approximated by a Nakagami distribution, and in which its total signal power Ω is modulated by a gamma distribution. The third distribution is called the Nakagami-generalized inverse Gaussian (NGIG) distribution (Agrawal and Karmeshu 2006), and it corresponds to a model in which the (approximating) Nakagami distribution has its total signal power Ω modulated by a generalized inverse Gaussian (GIG) distribution instead of a gamma distribution.

As shown in Destrempes and Cloutier (2010), none of these five other models is compatible with the limit case of a vanishing diffuse signal power $2\sigma^2\alpha_s$. Indeed, in that case, one should obtain an infinite intensity SNR (if $\varepsilon > 0$). It turns out that only the homodyned K-distribution and its related distributions satisfy that property. In fact, keeping α_s (the scattering cross sections homogeneity) and N_s (the number of random scatterers within the resolution cell) constant (see 10.3.2), one must have $\sigma^2 \to 0$ if the diffuse signal power $2\sigma^2\alpha_s N_s$ vanishes. Then, as computed in Destrempes and Cloutier (2010), one obtains an infinite intensity SNR if $\varepsilon > 0$, either for the Rice distribution or the homodyned K-distribution. Moreover, it was shown in Destrempes and Cloutier (2010) that the total signal power depends only on the coherent component in that case, which is a desirable property. Since the other distributions do not have these two properties, it makes the interpretation of their parameters more delicate, even if goodness-of-fit tests with data might be satisfying.

Finally, let us show that the Nakagami distribution also has these two properties. For that purpose, we consider a homodyned K-distribution with parameters ε, σ^2, and α, and its approximating Nakagami distribution with parameters $m = m(\varepsilon^2/(2\sigma^2), \alpha)$ and $\Omega = \varepsilon^2 + 2\sigma^2\alpha$ as at the beginning of this section. If $\varepsilon > 0$ and α are fixed and $\sigma^2 \to 0$, then, the parameter $\gamma = \varepsilon^2/(2\sigma^2) \to \infty$. From Theorem 26, part b), one has $\lim_{\gamma\to\infty} M_{\mathrm{HK}}^{(1)}(\gamma, \alpha) = 1$ (α being fixed). Thus, from Theorem 31, part b), one concludes that $m(\gamma, \alpha) \to \infty$. Therefore, one obtains $SNR = \sqrt{m(\gamma, \alpha)} \to \infty$. Moreover, if $\sigma^2 \to 0$, then $\Omega = \varepsilon^2 + 2\sigma^2\alpha \to \varepsilon^2$, which is independent of the diffuse signal parameters σ^2 and α. Therefore, the Nakagami, just as the Rice distribution and the homodyned K-distribution, is compatible with the limit case of a vanishing diffuse signal.

10.4 Parameter Estimation Methods

We discuss various known methods for the estimation of the parameters of the distributions presented in 10.3 based on an independent and identically distributed (i.i.d.) sample set $(A_1, \ldots, A_N)$ of positive real numbers (representing the amplitude).

10.4.1 Overview of a Few Estimation Methods

10.4.1.1 The MLE and the MAP

The MLE is defined as a critical point of the log-likelihood function (Edgeworth 1908, 1909; Fisher 1912, 1922, 1925) (the reader may also consult Pratt (1976))

$$L(\theta) = \sum_{i=1}^{N} \log P(A_i \mid \theta), \tag{10.12}$$

where θ represents the vector of parameters of the distribution and $\{A_1, \ldots, A_N\}$ is the sample data of size N. Actually, there might be multiple critical points and no global maximum (on the entire parameter domain). However, if the true value of the parameters is in the interior of a compact subset of the parameter domain, then the global maximum of the log-likelihood on that compact set converges to the true value of the parameters as the size of the sample tends to infinity (Redner 1981). Thus, one can define the MLE as the critical point with largest log-likelihood value (Redner and Walker 1984). A major difficulty lies in the analysis of the critical points: how many are there and which one coincides with the MLE? In fact, if the sample size is not sufficiently large, there might be no critical point of the log-likelihood function. Thus, one needs to address this issue before applying any numerical method to find the MLE.

One may also wish to impose a prior $\pi(\theta)$ on the parameters of the distribution. In that case, one considers the critical points of the constrained log-likelihood function

$$L(\theta) + \log \pi(\theta), \tag{10.13}$$

where $\log \pi(\theta)$ is viewed as a regularizing term. The MAP can then be defined as the critical point with largest value of the constrained log-likelihood function.

10.4.1.2 Moments Based Methods

Moments' methods have the advantage, over the MLE, of providing simpler and faster algorithms. On the other hand, the resulting systems of equations do not always admit a solution.

The simplest of these methods is based on the first few moments of the intensity. The number of moments considered is then equal to the number of parameters in the estimated distribution: one for the Rayleigh distribution, two for the Rice distribution, the K-distribution or the Nakagami distribution, and three for the homodyned K-distribution. Thus, one solves the system of equations

$$E[I^{\nu}] = \overline{I^{\nu}}, \quad \nu = 1, \ldots, r \tag{10.14}$$

where r is the number of parameters of the distribution. In Eq. (10.14), the left-hand side $E[I^{\nu}]$ represents a function of the parameters of the distribution, whereas the right-hand side $\overline{I^{\nu}}$ is the empirical moment computed from the data.

A slightly more complex method is based on the first few moments of the amplitude. Thus, one solves the system of equations

$$E[A^{\nu}] = \overline{A^{\nu}}, \quad \nu = 1, \ldots, r. \tag{10.15}$$

Since the intensity is the square of the amplitude, such methods use lower orders of the intensity, and thus, are likely to be less sensitive to noise. On the other hand, the analytical expressions of those moments are typically more complex than integral order moments of the intensity.

One may also use arbitrary fractional order moments of the intensity. For later reference, we find convenient to introduce the $M^{(\nu)}$-statistics, defined as

$$M^{(\nu)} = \frac{\overline{A^{\nu}}}{(\overline{I})^{\nu/2}}, \tag{10.16}$$

where $\nu > 0$. This statistic is the fractional moment of order ν of the amplitude normalized so that it becomes invariant under multiplication of the signal by a positive scaling constant.

Lemma 1 *For any non-constant random variable,* $0 < M^{(v)} < 1$, *if* $0 < v < 2$, *whereas* $M^{(v)} > 1$, *if* $v > 2$.

Proof If $v < 2$, the function $I^{v/2}$ is convex. Therefore, by Jensen's inequality (Jensen 1906), we have $E[A^v] = E[I^{v/2}] < (E[I])^{v/2}$ since the random variable I is non-constant. If $v > 2$, the function $I^{v/2}$ is concave and we obtain the reversed inequality. ■

If the number of parameters r is at least 2, the method based on the first few moments of the intensity is equivalent to solving the system

$$E[I] = \bar{I}; \qquad \frac{E[A^v]}{(E[I])^{v/2}} = \frac{\overline{A^v}}{(\bar{I})^{v/2}}, \quad v = 4, 6 \tag{10.17}$$

and thus amounts to working with the $M^{(4)}$ and $M^{(6)}$ statistics. Similarly, the method based on the first few moments of the amplitude is equivalent to solving the system

$$E[I] = \bar{I}; \qquad \frac{E[A^v]}{(E[I])^{v/2}} = \frac{\overline{A^v}}{(\bar{I})^{v/2}}, \quad v = 1, 3 \tag{10.18}$$

and thus amounts to working with the $M^{(1)}$ and $M^{(3)}$ statistics.

One may also combine various moments in the form of the SNR of a fractional order of the amplitude

$$R^{(v)} = \frac{\overline{A^v}}{(\overline{A^{2v}} - (\overline{A^v})^2)^{1/2}}, \tag{10.19}$$

or the skewness

$$S^{(v)} = \frac{\overline{A^{3v}} - 3\overline{A^v}\,\overline{A^{2v}} + 2(\overline{A^v})^3}{(\overline{A^{2v}} - (\overline{A^v})^2)^{3/2}}, \tag{10.20}$$

or the kurtosis

$$K^{(v)} = \frac{\overline{A^{4v}} - 4\overline{A^v}\,\overline{A^{3v}} + 6\overline{A^{2v}}\,(\overline{A^v})^2 - 3(\overline{A^v})^4}{(\overline{A^{2v}} - (\overline{A^v})^2)^2}. \tag{10.21}$$

Note that these three statistics can be expressed in terms of the family of $M^{(v)}$-statistics. For instance, we have $R^{(v)} = \frac{M^{(v)}}{(M^{(2v)} - (M^{(v)})^2)^{1/2}}$.

Considering more equations than the number of parameters of the distribution yields an overdetermined system of (non-linear) equations that may be solved in the sense of the least mean square (LMS). Thus, overall, all these methods amount to considering various combinations of the $M^{(v)}$-statistics.

10.4.1.3 Log-Moments Based Methods

One may also work with moments of functions of the intensity that involve its logarithm. Ideally, one would consider powers of the logarithm of the intensity. But powers greater than 1 appear to be intractable for the distributions considered in this chapter, because the resulting integrals are not known explicitly as far as we can tell. Moreover, one may want to obtain functions that are invariant under a change of the intensity by a scaling factor. Thus, one is led to the so-called U-statistics (Oliver 1993)

$$U = \overline{\log I} - \log \overline{I}, \tag{10.22}$$

and the X-statistics (Blacknell and Tough 2001)

$$X = \overline{I \log I}/\overline{I} - \overline{\log I}. \tag{10.23}$$

Lemma 2 *For any non-constant random variable,* $U < 0$.

Proof The function $\log I$ is convex. Therefore, from Jensen's inequality, we obtain $E[\log I] < \log E[I]$, since the random variable I is non-constant. ■

Lemma 3 *For any non-constant random variable,* $X > 0$.

Proof The function $I \log I$ is concave. Thus, $E[I \log I] > E[I] \log E[I]$. From Lemma 2, we conclude that $E[I \log I] > E[I]E[\log I]$. ■

10.4.2 Parameter Estimation Method for the Rayleigh Distribution

Since a Rayleigh distribution with parameter σ^2 on the amplitude A is equivalent to an exponential distribution with parameter $2\sigma^2$ on the intensity $I = A^2$, the MLE of the parameter σ^2 is equal to $\overline{I}/2$. Note that, in this special case, the MLE coincides with the estimator based on the first moment of the intensity.

10.4.3 Parameter Estimation Methods for the Rice Distribution

10.4.3.1 The MLE for the Rice Distribution

In Talukdar and Lawing (1991), the Rice distribution is estimated in the sense of the MLE, as follows.

Theorem 5 (Talukdar and Lawing 1991). *Let $A_1, \ldots, A_N$ be a finite sample set of positive numbers. Let $\varepsilon \geq 0$ and $\sigma^2 > 0$ be the parameters of the Rice distribution. Let $\mu = \varepsilon^2 + 2\sigma^2$, and $\kappa = \varepsilon^2/(2\sigma^2)$. Let $y_i = A_i/\sqrt{\bar{I}}$, where $\bar{I} = 1/N \sum_{i=1}^{N} A_i^2$. Then, the critical points of the log-likelihood function $L_{Ri}(\varepsilon, \sigma^2)$ of the Rice distribution are the points of the form*

$$\varepsilon = \sqrt{\mu\kappa/(\kappa+1)}; \quad \sigma^2 = \mu/(2(\kappa+1)), \tag{10.24}$$

where $\mu = \bar{I}$ *and $\kappa \geq 0$ is any root of the function $f(\kappa)$ defined by*

$$\frac{1}{(1+\kappa)} + \frac{(1+2\kappa)}{N\sqrt{\kappa(1+\kappa)}} \sum_{i=1}^{N} y_i \frac{I_1(2y_i\sqrt{\kappa(1+\kappa)})}{I_0(2y_i\sqrt{\kappa(1+\kappa)})} - 2. \tag{10.25}$$

Here, I_p denotes the modified Bessel function of the first kind of order p (the subscript avoids the confusion with the intensity I).

Theorem 5 gives useful information on the value of μ for the critical points of the log-likelihood function. It also introduces a one-variable function $f(\kappa)$. But the main drawback is the lack of information on the roots of f. Fortunately, a more recent result gives complete information about the critical points of the log-likelihood function of the Rice distribution in the following form.

Theorem 6 (Carrobi and Cati 2008). *Let $A_1, \ldots, A_N$ be a finite sample set of positive numbers. Let $\varepsilon \geq 0$ and $\sigma^2 > 0$ be the parameters of the Rice distribution. Let $\bar{I} = 1/N \sum_{i=1}^{N} A_i^2$. Assume that the elements A_i are not all identical. Then, the log-likelihood function $L_{\mathrm{Ri}}(\varepsilon, \sigma^2)$ of the Rice distribution has exactly two critical points: $(0, \bar{I}/2)$ and another one, denoted $(\hat{\varepsilon}, \hat{\sigma}^2)$, that satisfies $\hat{\varepsilon} > 0$ and $\hat{\sigma}^2 > 0$. Moreover, the MLE is the second one. In fact, the MLE is actually an absolute maximum of the log-likelihood function on its domain.*

Thus, the MLE $(\hat{\varepsilon}, \hat{\sigma}^2)$ consists of the unique critical point of the log-likelihood function $L_{\mathrm{Ri}}(\varepsilon, \sigma^2)$ for which both coordinates are positive.

10.4.3.2 Expression of Fractional Order Moments of the Amplitude

The $M^{(\nu)}$-statistics is explicitly known for the Rice distribution.

Theorem 7 (Rice 1954). *Assume that $A = \sqrt{I}$ is distributed according to the Rice distribution, with parameters $\varepsilon \geq 0$* and $\sigma^2 > 0$. *Set $\kappa = \varepsilon^2/(2\sigma^2)$. Then, the $M^{(\nu)}$-statistics $E[A^\nu]/E[I]^{\nu/2}$ is equal to*

$$M_{\mathrm{Ri}}^{(\nu)}(\kappa) = \frac{\Gamma(\nu/2+1)}{(\kappa+1)^{\nu/2}} \, {}_1F_1(-\nu/2, 1, -\kappa), \tag{10.26}$$

where ${}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z)$ *is the generalized hypergeometric series (here* $p = q = 1$*).*

10.4.3.3 Method Based on the Moments of the Amplitude

In Talukdar and Lawing (1991), an estimation method of the Rice distribution based on the first two moments of the amplitude is proposed as an alternative to the MLE. The method consists of solving the system of equations

$$E[A] = \overline{A}; \qquad E[A^2] = \overline{A^2}. \tag{10.27}$$

For that purpose, it is proposed to consider the equivalent system $E[I] = \overline{I}$ and $\frac{E[A]}{E[I]^{1/2}} = M^{(1)}$. The point of using this equivalent system is that the $M^{(1)}$-statistics for the Rice distribution depends only on the parameter κ. As a special case of Theorem 7, we have the $M^{(1)}$-statistics.

Corollary 1 Talukdar and Lawing (1991). *Assume that* $A = \sqrt{I}$ *is distributed according to the Rice distribution, with parameters* $\varepsilon \geq 0$ *and* $\sigma^2 > 0$. *Set* $\kappa = \varepsilon^2/(2\sigma^2)$. *Then, the* $M^{(1)}$*-statistics* $E[A]/E[I]^{1/2}$ *is equal to*

$$M^{(1)}_{\mathrm{Ri}}(\kappa) = \frac{\Gamma(3/2)}{\sqrt{\kappa + 1}} e^{-\kappa/2}\Big((1 + \kappa)I_0(\kappa/2) + \kappa I_1(\kappa/2)\Big), \tag{10.28}$$

where I_p *denotes the modified Bessel function of the first kind of order* p.

10.4.3.4 Discussion

In this section, we present a new result on the computation of the MLE of the Rice distribution. We show that the Talukdar-Lawing estimator of Sect. 10.4.3.3 can be computed with a binary search algorithm. We introduce two log-moments based methods for the Rice distribution. Finally, we compare these estimators on simulated data.

Concerning the MLE computation, a little more work allows to combine Theorems 5 and 6 into the following result. See Fig. 10.2 for an illustration of the function $f(\kappa)$.

Theorem 8 *Notation as in Theorem 5. Assume that the data elements* A_i *are not all identical. Then, the function* $f(\kappa)$ *of Eq.* 10.25 *has exactly two non-negative roots:* 0 *and a unique positive root, denoted* κ_*. *The MLE is expressed as in Eq.* (10.24), *with* $\kappa = \kappa_*$ *(i.e. the unique positive root of the function* f*). Moreover,* $f(\kappa) > 0$ *on the interval* $(0, \kappa_*)$, *and* $f(\kappa) < 0$ *on the interval* (κ_*, ∞).

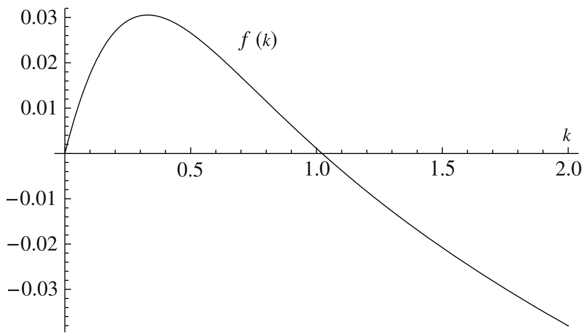

Fig. 10.2 Typical behavior of the function $f(\kappa)$ defined in Eq. (10.25), assuming that the data elements are not all identical. The function f has two non-negative roots (0 and $\kappa_* > 0$); $f(\kappa) > 0$ for $0 < \kappa < \kappa_*$, and $f(\kappa) < 0$ for $\kappa > \kappa_*$ (here, $N = 2$, $y_1 = 0.5$, $y_2 = \sqrt{2-(0.5)^2}$, so that $\frac{1}{2}(y_1^2 + y_2^2) = 1$)

Theorem 8 implies that an efficient binary search algorithm can be used for the computation of the MLE of the Rice distribution.

Concerning the estimation method based on the $M^{(v)}$-statistics, in general, there is no closed form for a solution to the equation. $M_{\mathrm{Ri}}^{(v)}(\kappa) = M$, but one can use the following result, relevant for any $v \neq 2$. See Fig. 10.3, right column, for an illustration of the function $M_{\mathrm{Ri}}^{(v)}(\kappa)$.

Theorem 9 *Let* $v > 0$. *Then,*

(a) $\lim_{\kappa \to 0} M_{\mathrm{Ri}}^{(v)}(\kappa) = \Gamma(v/2 + 1)$;

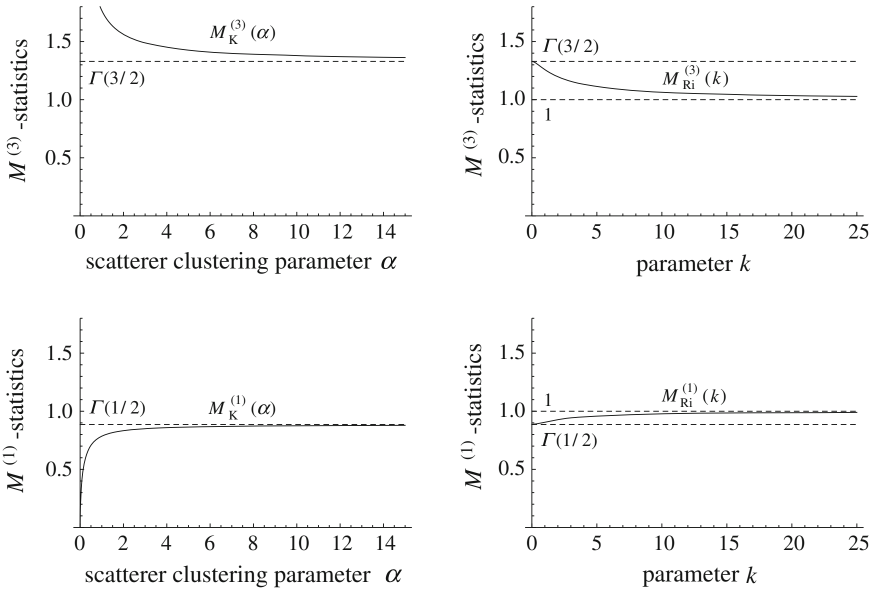

Fig. 10.3 Typical behavior of the $M^{(v)}$-statistics for the K-distribution (*left column*) and the Rice distribution (*right column*), when $v < 2$ (*bottom row*) and $v > 2$ (*top row*)

(b) $\lim_{\kappa\to\infty} M_{\mathrm{Ri}}^{(v)}(\kappa) = 1$;

(c) *if* $v<2$, *the function* $M_{\mathrm{Ri}}^{(v)}(\kappa)$ *is an increasing function; if* $v>2$, *the function* $M_{\mathrm{Ri}}^{(v)}(\kappa)$ *is a decreasing function.*

Thus, let $M>0$ be a real number (playing the role of the $M^{(v)}$-statistics). If $v<2$ and $\Gamma(v/2+1)\le M<1$, then an efficient binary search algorithm yields the unique solution to the equation $M_{\mathrm{Ri}}^{(v)}(\kappa)=M$. Indeed, from Theorem 9, the function $M_{\mathrm{Ri}}^{(v)}(\kappa)$ is increasing in that case and its range is the interval $[\Gamma(v/2+1),1)$. On the other hand, if $v<2$ and $M<\Gamma(v/2+1)$, then there is no solution to the equation $M_{\mathrm{Ri}}^{(v)}(\kappa)=M$. Nevertheless, in that case, the value $\kappa=0$ minimizes the distance between $M_{\mathrm{Ri}}^{(v)}(\kappa)$ and M. Thus, it makes sense to take $\kappa=0$. Similarly, if $v>2$ and $1<M\le\Gamma(v/2+1)$, then there is a unique solution to the equation $M_{\mathrm{Ri}}^{(v)}(\kappa)=M$, and this solution can be found efficiently with a binary search algorithm. On the other hand, if $v>2$ and $M>\Gamma(v/2+1)$, one may take $\kappa=0$. Thus, it makes sense to switch to the Rayleigh model (corresponding to $\kappa=0$), whenever the equation $M_{\mathrm{Ri}}^{(v)}(\kappa)=M$ has no solution. This argument applies to the special case where $v=1$, which corresponds to the Talukdar-Lawing method of Corollary 1. For later reference, we introduce here what we call the Rice conditions

$$v<2 \text{ and } \Gamma(v/2+1)\le M<1, \text{ or } v>2 \text{ and } 1<M\le\Gamma(v/2+1). \tag{10.29}$$

Thus, as explained above, the equation $M_{\mathrm{Ri}}^{(v)}(\kappa)=M$ has a solution if and only if the Rice conditions are satisfied. Note that the U and X-statistics can be computed analytically for the Rice distribution.

Theorem 10 *Assume that* $A=\sqrt{I}$ *is distributed according to the Rice distribution, with parameters* $\varepsilon\ge 0$ *and* $\sigma^2>0$. *Set* $\kappa=\varepsilon^2/(2\sigma^2)$. *Then,*

(a) *the U-statistics* $E[\log I]-\log E[I]$ *is equal to*

$$U_{\mathrm{Ri}}(\kappa)=\Gamma(0,\kappa)+\log\frac{\kappa}{\kappa+1}, \tag{10.30}$$

where $\Gamma(0,x)$ *is the incomplete gamma function* $\int_x^\infty \frac{e^{-t}}{t}\,dt$;

(b) *the X-statistics* $E[I\log I]/E[I]-E[\log I]$ *is equal to*

$$X_{\mathrm{Ri}}(\kappa)=\frac{1}{\kappa+1}(2-e^{-\kappa}). \tag{10.31}$$

Theorem 11 below shows that a binary search algorithm can be used to solve the equation $U_{\mathrm{Ri}}=U$ if and only if $U\ge-\gamma_E$, where γ_E is the Euler's constant. If ever $U<-\gamma_E$, one may switch to the Rayleigh model ($\kappa=0$). Similarly, Theorem 12 shows that the equation $X_{\mathrm{Ri}}=X$ has a solution (which is then unique

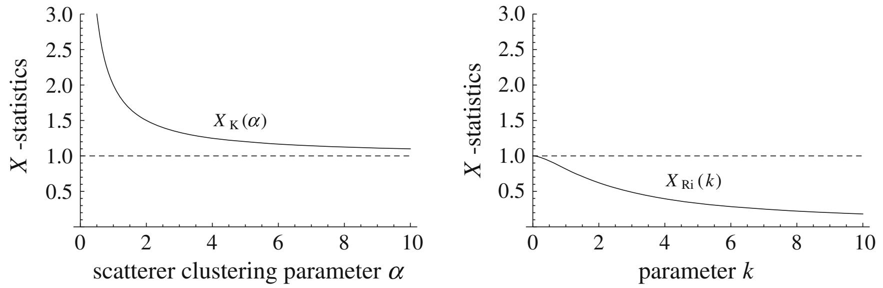

Fig. 10.4 Typical behavior of the X-statistics for the K-distribution (*left column*) and the Rice distribution (*right column*)

and can be found with a binary search algorithm) if and only if $X < 4$. If ever $X \geq 4$, one may switch to the Rayleigh model.

Theorem 11

(a) $\lim_{\kappa \to 0} U_{\mathrm{Ri}}(\kappa) = -\gamma_E$, *where γ_E is the Euler's constant;*
(b) $\lim_{\kappa \to \infty} U_{\mathrm{Ri}}(\kappa) = 0$;
(c) *the function $U_{\mathrm{Ri}}(\kappa)$ is an increasing function.*

Theorem 12

(a) $\lim_{\kappa \to 0} X_{\mathrm{Ri}}(\kappa) = 1$;
(b) $\lim_{\kappa \to \infty} X_{\mathrm{Ri}}(\kappa) = 0$;
(c) *the function $X_{\mathrm{Ri}}(\kappa)$ is a decreasing function.*

See Fig. 10.4, right column, for an illustration of the X-statistics for the Rice distribution.

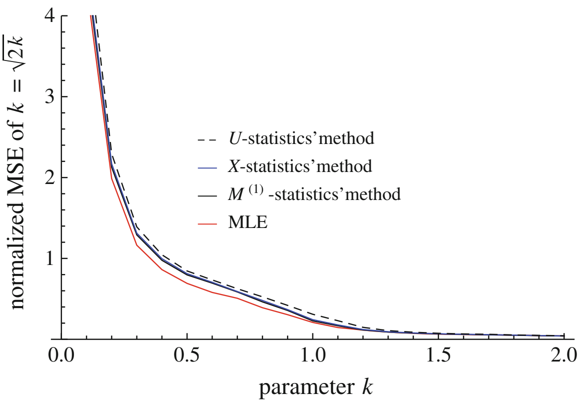

Fig. 10.5 Comparison of the normalized MSE $\sqrt{E[(\hat{k} - k)^2]}/k$, where $k = \varepsilon/\sigma$, for the estimators based on the MLE (*red solid line*), the $M^{(1)}$-statistics (*black solid line*), the X-statistics (*blue solid line*) and the U-statistics (*dashed line*) in the case of the Rice distribution. The sample size is $N = 1{,}000$

In order to compare these four estimators, we considered the parameter $k = \sqrt{2\kappa} = \varepsilon/\sigma$ with values in the set $\{0.1, 0.2, \ldots, 2.0\}$. For each value of k, 1,000 datasets of $N = 1,000$ elements each were simulated according to the corresponding Rice distribution. Thus, we could estimate the normalized mean squared error (MSE) of the estimator $\hat{k}$ as $\sqrt{E[(\hat{k}-k)^2]}/k$. The resulting normalized MSE curves are presented in Fig. 10.5. As one can see, the MLE is slightly better than the estimators based on the $M^{(1)}$ or the X-statistics. The method based on the U-statistics is slightly worse than the other estimators. The two estimators based on the $M^{(1)}$ and the X-statistics are practically equivalent.

10.4.4 Parameter Estimation Methods for the K-Distribution

10.4.4.1 The MLE for the K-Distribution

The partial derivatives of the log-likelihood function of the K-distribution with respect to α and σ^2 are equal to

$$\frac{\partial}{\partial\alpha} L_{\mathrm{K}}(\sigma^2, \alpha) = -N\psi(\alpha) + \sum_{i=1}^{N} \log\left(\frac{1}{\sqrt{2\sigma^2}} A_i\right) + \frac{\frac{\partial}{\partial\alpha} K_{\alpha-1}\left(\sqrt{\frac{2}{\sigma^2}} A_i\right)}{K_{\alpha-1}\left(\sqrt{\frac{2}{\sigma^2}} A_i\right)}; \qquad (10.32)$$

$$\frac{\partial}{\partial\sigma^2} L_{\mathrm{K}}(\sigma^2, \alpha) = -\frac{N\alpha}{\sigma^2} + \sum_{i=1}^{N} \frac{1}{\sigma^2}\left(\frac{1}{\sqrt{2\sigma^2}} A_i\right) \frac{K_{\alpha}\left(\sqrt{\frac{2}{\sigma^2}} A_i\right)}{K_{\alpha-1}\left(\sqrt{\frac{2}{\sigma^2}} A_i\right)}. \qquad (10.33)$$

Solutions to this system of two non-linear equations are found numerically in Joughin et al. (1993). In Roberts and Furui (2000), an Expectation-Maximization (EM) algorithm is proposed for finding the MLE. In that context, the variable w of Eq. (10.4) is viewed as the latent variable. A variant of the EM algorithm is used in Chung et al. (2005) in place of the standard EM algorithm.

However, none of the methods (Joughin et al. 1993; Roberts and Furui 2000; Chung et al. 2005) can be used in full generality, because the MLE is not always well-defined for the K-distribution. See Sect. 10.4.4.6 for a discussion on that issue.

10.4.4.2 Expression of Fractional Order Moments of the Amplitude

The $M^{(\nu)}$-statistics is explicitly known for the K-distribution.

Theorem 13 (Dutt and Greenleaf 1995). *Assume that* $A = \sqrt{I}$ *is distributed according to the K-distribution, with parameters* $\sigma^2 > 0$ and $\alpha > 0$. *Then, the* $M^{(v)}$*-statistics* $E[A^v]/E[I]^{v/2}$ *is equal to*

$$M_{\mathrm{K}}^{(v)}(\alpha) = \Gamma(v/2+1)\frac{\Gamma(v/2+\alpha)}{\alpha^{v/2}\Gamma(\alpha)}. \tag{10.34}$$

10.4.4.3 A Method Based on the Moments of the Intensity

The simplest moments method consists in solving the system of equations

$$E[I] = \overline{I}; \qquad E[I^2] = \overline{I^2}. \tag{10.35}$$

Equivalently, that method is based on the mean intensity and the $M^{(4)}$-statistics (that statistics is called the V-statistics in Blacknell and Tough (2001)). One computes for the K-distribution $V_{\mathrm{K}}(\alpha) = E[I^2]/E[I]^2 = 2\left(1+\frac{1}{\alpha}\right)$. Thus, there is a solution to the system (10.35) if and only if $V > 2$, in which case the solution is equal to $\alpha = 2/(V-2)$.

10.4.4.4 Two Methods Based on Fractional Order Moments of the Amplitude

In Dutt and Greenleaf (1995), the authors suggest to use the SNR based on fractional order moments in the form of the $R^{(v)}$-statistics, where $v > 0$. In that study, it is shown that a value of $v = 1/4$ yields a reliable estimator. We have the following result (note that there is a typographical error in Dutt and Greenleaf (1995, Eq. (6), p. 253)).

Theorem 14 (Dutt and Greenleaf 1995). *Assume that A is distributed according to the K-distribution, with parameters* $\sigma^2 > 0$ and $\alpha > 0$. *Then, the* $R^{(v)}$*-statistics* $\frac{E[A^v]}{\sqrt{E[A^{2v}] - E[A^v]^2}}$ *is expressed as*

$$R_{\mathrm{K}}^{(v)}(\alpha) = \frac{\Gamma(v/2+1)\Gamma(v/2+\alpha)}{\sqrt{\Gamma(v+1)\Gamma(v+\alpha)\Gamma(\alpha) - \Gamma^2(v/2+1)\Gamma^2(v/2+\alpha)}}. \tag{10.36}$$

In Iskander and Zoubir (1999), the authors suggest the use of fractional order moments in the form of the Y-statistics $\frac{E[A^{2r+2s}]}{E[A^{2r}]E[A^{2s}]}$, where $s > 0$, and $r \in \mathbb{N}$. It is shown that a value of $s < 2$ yields lower variance of the resulting estimator, taking $r = 1$. The following result holds.

Theorem 15 (Iskander and Zoubir 1999). *Assume that A is distributed according to the K-distribution, with parameters $\sigma^2 > 0$ and $\alpha > 0$. Then, the Y-statistics $\frac{E[A^{2+2s}]}{E[A^2]E[A^{2s}]}$ is expressed as*

$$Y_{\mathrm{K}}(\alpha) = (1+s)(1+\frac{s}{\alpha}). \tag{10.37}$$

Using Theorem 15, there is a solution to the equation $Y_{\mathrm{K}}(\alpha) = Y$ if and only if $Y > 1 + s$. In that case, $\alpha = \frac{s(1+s)}{Y-(1+s)}$ is the unique solution. Note that the V-statistics corresponds to the special case where $s = 1$.

10.4.4.5 Two Log-Moments Methods

In the case of the K-distribution, it has been proposed (Oliver 1993) to use the U-statistics in order to estimate α.

Theorem 16 (Oliver 1993). *Assume that $\sqrt{I}$ is distributed according to the K-distribution, with parameters $\sigma^2 > 0$* and *$\alpha > 0$. Then, the U-statistics $E[\log I] - \log E[I]$ is expressed as*

$$U_{\mathrm{K}}(\alpha) = -\gamma_E + \psi(\alpha) - \log \alpha, \tag{10.38}$$

where γ_E is the Euler's constant and $\psi(z) = d[\log \Gamma(z)]/dz$ is the digamma function (Abramowitz and Stegun 1972, (6.3.1)).

There is also a method (Blacknell and Tough 2001) based on the X-statistics.

Theorem 17 (Blacknell and Tough 2001). *Assume that $\sqrt{I}$ is distributed according to the K-distribution, with parameters $\sigma^2 > 0$* and *$\alpha > 0$. Then, the X-statistics $E[I \log I]/E[I] - E[\log I]$ is expressed as*

$$X_{\mathrm{K}}(\alpha) = 1 + \frac{1}{\alpha}. \tag{10.39}$$

Lemma 3 guarantees that X is non-negative. Thus, there is a solution to the equation $X_{\mathrm{K}}(\alpha) = X$ if and only if $X > 1$, in which case the unique solution is equal to $\alpha = 1/(X-1)$. See Fig. 10.4, left column, for an illustration of the X-statistics for the K-distribution.

10.4.4.6 Discussion

In this section, we present further results on the MLE and the MAP of the K-distribution. We then show that the methods introduced in Sects. 10.4.4.3 to 10.4.4.5 can be solved with a binary search algorithm. Finally, we present a comparison of these estimators on simulated data.

Arguing that the existing methods for computing the MLE are time consuming and that moments based methods do not always lead to a solution of the resulting equations, a Bayesian estimation method of the SNR (denoted D and called the detection index) was proposed in Abraham and Lyons (2010).

To clarify the notion of MLE for the K-distribution, we present the following two results.

Theorem 18 *Let* $\alpha > 0$ *be fixed. Then, there exists a root* $\sigma^2(\alpha, \tilde{A}) > 0$ *of* $\frac{\partial}{\partial \sigma^2} L_{\mathrm{K}}(\sigma^2, \alpha)$.

Theorem 19 *Let* $N \geq 1$ *be the sample size and* $\sigma^2(\alpha, \tilde{A})$ *denote any root of* $\frac{\partial}{\partial \sigma^2} L_{\mathrm{K}}(\sigma^2, \alpha)$. *Then,*

(a) $\lim_{\alpha \to 0} \alpha \frac{\partial}{\partial \alpha} L_{\mathrm{K}}(\sigma^2(\alpha, \tilde{A}), \alpha) \geq N$.
(b) $\lim_{\alpha \to \infty} \alpha \frac{\partial}{\partial \alpha} L_{\mathrm{K}}(\sigma^2(\alpha, \tilde{A}), \alpha) = 0$.

Thus, if ever the function $\frac{\partial}{\partial \alpha} L_{\mathrm{K}}(\sigma^2(\alpha, \tilde{A}), \alpha)$ is decreasing for some sample set, then there is no MLE. This is the case, for instance, if $\{A_1, A_2\} = \{\frac{1}{2}, \frac{\sqrt{7}}{2}\}$ (see Fig. 10.6). So, the point in considering other estimators than the MLE is not so much that its computation is time consuming, but rather that it is not always well-defined for the K-distribution.

However, one may set a prior on the parameters σ^2 and α and see if the maximum a posteriori (MAP) is well-defined. For the K-distribution, let us consider the prior $\pi(\alpha) = 1/\alpha$ (so, this prior does not depend on σ^2 for simplicity of technical considerations). This amounts to setting the Jeffreys non-informative prior (Jeffreys 1946) on the parameter α. Recall that the Jeffreys prior is defined as

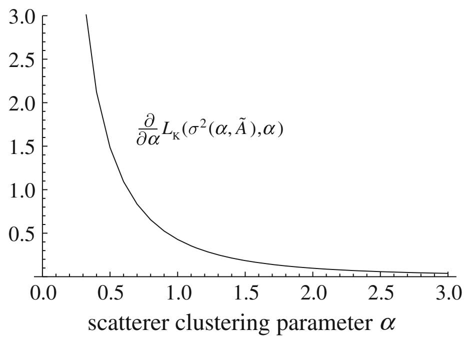

Fig. 10.6 Behavior of the function $\frac{\partial}{\partial \alpha} L_{\mathrm{K}}(\sigma^2(\alpha, \tilde{A}), \alpha)$ for the K-distribution in the case where $\tilde{A} = \{\frac{1}{2}, \frac{\sqrt{7}}{2}\}$. The MLE is not defined in this case

$\pi(\alpha) = \left(I_F(\alpha)\right)^{1/2}$, where $I_F(\alpha)$ denotes the Fisher information (Fisher 1956), namely $I_F(\alpha) = -E[\frac{\partial^2}{\partial^2\alpha}\log P(A\,|\,\sigma^2,\alpha)] = E[\left(\frac{\partial}{\partial\alpha}\log P(A\,|\,\sigma^2,\alpha)\right)^2]$. In Abraham and Lyons (2010), it is shown that $\pi(\alpha)\sim 1/\alpha$ for large values of α. Then, the MAP corresponds to a solution to the system of equations

$$\frac{\partial}{\partial\alpha}L_{\mathrm{K}}(\sigma^2,\alpha) + \frac{\partial}{\partial\alpha}\log\pi(\alpha) = 0; \tag{10.40}$$

$$\frac{\partial}{\partial\sigma^2}L_{\mathrm{K}}(\sigma^2,\alpha) = 0. \tag{10.41}$$

Now, with the proposed prior, we obtain $\frac{\partial}{\partial\alpha}\log\pi(\alpha) = -1/\alpha$. Then, from Theorem 19, we know that $\frac{\partial}{\partial\sigma^2}L_{\mathrm{K}}(\sigma^2(\alpha,\tilde{A}),\alpha) - 1/\alpha \geq (N-1)/\alpha > 0$, for α sufficiently small and $N>1$, and that $\frac{\partial}{\partial\alpha}L_{\mathrm{K}}(\sigma^2(\alpha,\tilde{A}),\alpha) - 1/\alpha < 0$ for α sufficiently large. Therefore, the Intermediate Value Theorem implies that there is $\alpha>0$ for which $\frac{\partial}{\partial\alpha}L_{\mathrm{K}}(\sigma^2(\alpha,\tilde{A}),\alpha) - 1/\alpha = 0$. Thus, this MAP estimator is well-defined for the K-distribution. Furthermore, its computation is amenable to a binary search algorithm. Note that one may have chosen the prior $\pi(\sigma^2,\alpha) = \frac{1}{\sigma}$ because it is *scale-invariant* (i.e., $P_K(A\,|\,\sigma^2,\alpha) = \frac{1}{\sigma}P_K(\frac{A}{\sigma}\,|\,1,\alpha)$). However, with that choice of prior, one may have an undefined MAP estimator. Other priors are possible, but we have not explored that avenue here.

Note that in Abraham and Lyons (2010), it is advocated to take the non-informative prior $1/\alpha^2$ instead of the Jeffreys prior $1/\alpha$, in order to obtain a posterior distribution with a well-defined mean, i.e. such that $\int_0^\infty \alpha P(\alpha\,|\,\tilde{A})\,d\alpha < \infty$. But, there is no need to require a finite posterior mean to define the MAP. The only requirement is a finite sum for the posterior distribution (i.e., $\int_0^\infty P(\alpha\,|\,\tilde{A})\,d\alpha < \infty$. Now, taking the prior $1/\alpha^2$ (Abraham and Lyons 2010), the first statement is equivalent to $\int_0^\infty \alpha\prod_{i=1}^N P(A_i\,|\,\sigma^2\alpha)\frac{1}{\alpha^2}\,d\alpha < \infty$. On the other hand, taking the Jeffreys prior $1/\alpha$, the second statement is equivalent to $\int_0^\infty\prod_{i=1}^N P(A_i\,|\,\sigma^2\alpha)\frac{1}{\alpha}\,d\alpha < \infty$. Thus, as one can see, the two statements are equivalent (because, two different priors are considered).

One may wish to simplify the above MAP estimator by considering a hybrid MAP. Namely, the first moment of the intensity yields the identity $\sigma^2 = \bar{I}/(2\alpha)$. Substituting this expression into the difference of Eq. (10.40) with Eq. (10.41) yields the equation

$$\left\{\alpha\frac{\partial}{\partial\alpha}L_{\mathrm{K}}(\sigma^2,\alpha) - \sigma^2\frac{\partial}{\partial\sigma^2}L_{\mathrm{K}}(\sigma^2,\alpha)\right\}\Big|_{\sigma^2=\bar{I}/(2\alpha)} - 1 = 0. \tag{10.42}$$

The following result implies that a solution to Eq. (10.42) can be found with a binary search algorithm, provided that the sample size N is greater than 1.

Theorem 20 *Let $N \geq 1$ be the sample size. Then,*

a) $\lim_{\alpha\to 0}\left\{\alpha \frac{\partial}{\partial\alpha} L_{\mathrm{K}}(\sigma^2,\alpha) - \sigma^2 \frac{\partial}{\partial\sigma^2} L_{\mathrm{K}}(\sigma^2,\alpha)\right\}\Big|_{\sigma^2=\bar{I}/(2\alpha)} = N.$

b) $\lim_{\alpha\to \infty}\left\{\alpha \frac{\partial}{\partial\alpha} L_{\mathrm{K}}(\sigma^2,\alpha) - \sigma^2 \frac{\partial}{\partial\sigma^2} L_{\mathrm{K}}(\sigma^2,\alpha)\right\}\Big|_{\sigma^2=\bar{I}/(2\alpha)} = 0.$

Concerning the estimation method based on the $M^{(v)}$-statistics, in general, there is no closed form for a solution to the equation $M_{\mathrm{K}}^{(v)}(\alpha) = M$, but one can use the following result, relevant for any $v \neq 2$. See Fig. 10.3, left column, for an illustration of the function $M_{\mathrm{K}}^{(v)}(\alpha)$.

Theorem 21 *We have the following properties*

(a) *if* $v<2$, then $\lim_{\alpha\to 0} M_{\mathrm{K}}^{(v)}(\alpha) = 0$; *if* $v > 2$, *then* $\lim_{\alpha\to 0} M_{\mathrm{K}}^{(v)}(\alpha) = \infty$;
(b) $\lim_{\alpha\to \infty} M_{\mathrm{K}}^{(v)}(\alpha) = \Gamma(v/2+1)$;
(c) *if* $v<2$, then $M_{\mathrm{K}}^{(v)}(\alpha)$ *is strictly increasing on its domain* $(0,\infty)$; *if* $v > 2$, *then* $M_{\mathrm{K}}^{(v)}(\alpha)$ *is strictly decreasing on its domain* $(0,\infty)$.

So, let $M > 0$ be a real number (playing the role of the $M^{(v)}$-statistics). If $v<2$ and $0<M<\Gamma(v/2+1)$, then an efficient binary search algorithm yields the unique solution to the equation $M_{\mathrm{K}}^{(v)}(\alpha) = M$. Indeed, from Theorem 21, the function $M_{\mathrm{K}}^{(v)}(\alpha)$ is increasing in that case and its range is the interval $(0,\Gamma(v/2+1))$. On the other hand, if $v<2$ and $\Gamma(v/2+1) \leq M < 1$, then there is no solution to the equation $M_{\mathrm{K}}^{(v)}(\alpha) = M$. However, in that case, the distance between $M_{\mathrm{K}}^{(v)}(\alpha)$ and M is minimal as $\alpha \to \infty$. Thus, it makes sense to take the Rayleigh distribution. Similarly, if $v > 2$ and $M > \Gamma(v/2+1)$, then there is a unique solution to the equation $M_{\mathrm{K}}^{(v)}(\alpha) = M$, and this solution can be found efficiently with a binary search algorithm. On the other hand, if $v > 2$ and $1<M<\Gamma(v/2+1)$, one may take $\alpha \to \infty$. For later reference, we introduce here what we call the K-distribution conditions

$$v<2 \text{ and } 0<M<\Gamma(v/2+1), \text{ or } v > 2 \text{ and } \Gamma(v/2+1)<M. \tag{10.43}$$

Thus, the equation $M_{\mathrm{K}}^{(v)}(\alpha) = M$ has a solution if and only if the K-distribution conditions are satisfied. Note that the Rice conditions (10.29) and the K-distribution conditions (10.43) are mutually exclusive and they are exhaustive (that it is to say, with the understanding that M plays the role of the $M^{(v)}$-statistics).

Concerning the parameter estimation method (Dutt and Greenleaf 1995), the following result shows that the equation $R_{\mathrm{K}}^{(v)}(\alpha) = R$, where $R > 0$ plays the role of the $R^{(v)}$-statistics, has a solution if and only if $R < \frac{\Gamma(v/2+1)}{\sqrt{\Gamma(v+1)-\Gamma^2(v/2+1)}}$, and that there is at most one solution. Moreover, it shows that an efficient binary search algorithm can be used to find the solution, whenever it exists. Finally, one sees that the solution $\alpha = \infty$ is the one that minimizes the distance between $R_{\mathrm{K}}^{(v)}(\alpha)$ and R, whenever the equation $R_{\mathrm{K}}^{(v)}(\alpha) = R$ has no solution. This amounts to switch to the Rayleigh model, with parameter $a^2 = \lim_{\alpha\to\infty} \sigma^2\alpha = \bar{I}$.

Theorem 22 *The following properties hold*

(a) $\lim_{\alpha\to 0} R_{\mathrm{K}}^{(v)}(\alpha) = 0$;
(b) $\lim_{\alpha\to\infty} R_{\mathrm{K}}^{(v)}(\alpha) = \frac{\Gamma(v/2+1)}{\sqrt{\Gamma(v+1)-\Gamma^2(v/2+1)}}$;
(c) $R_{\mathrm{K}}^{(v)}(\alpha)$ *is strictly increasing on its domain* $(0, \infty)$.

Concerning the method of Oliver (1993), the following result shows that an efficient binary search algorithm can be used in order to find the unique solution to the equation $U_{\mathrm{K}}(\alpha) = U$, whenever $U < -\gamma_E$. If ever $U \geq -\gamma_E$, Theorem 23 shows that it makes sense to adopt the Rayleigh model.

Theorem 23 *We have the following properties*

(a) $\lim_{\alpha\to 0} U_{\mathrm{K}}(\alpha) = -\infty$;
(b) $\lim_{\alpha\to\infty} U_{\mathrm{K}}(\alpha) = -\gamma_E$;
(c) $U_{\mathrm{K}}(\alpha)$ *is strictly increasing on its domain* $(0, \infty)$.

Similarly, one may switch to the Rayleigh distribution, whenever $V \leq 2$ (c.f. Sect. 10.4.4.3), or $Y \leq 1 + s$ (c.f. Sect. 10.4.4.4), or $X \leq 1$ (c.f. Sect. 10.4.4.5). Theorems 22 and 23 are illustrated in Fig. 10.7.

In order to compare the various estimators, we considered the parameter α with values in the set $\{1, 2, \ldots, 20\}$. For each value of α, $1,000$ datasets of $N = 1,000$ elements each were simulated according to the corresponding K-distribution. As in Dutt and Greenleaf (1994), one may consider the estimation of the parameter $\beta = 1/\alpha$ instead of α itself. In that case, one does not need to discard values of $1/\alpha$, because whenever the method has no solution, one may switch to the Rayleigh model ($\alpha = \infty$), which corresponds to $1/\alpha = 0$. Thus, we could estimate the

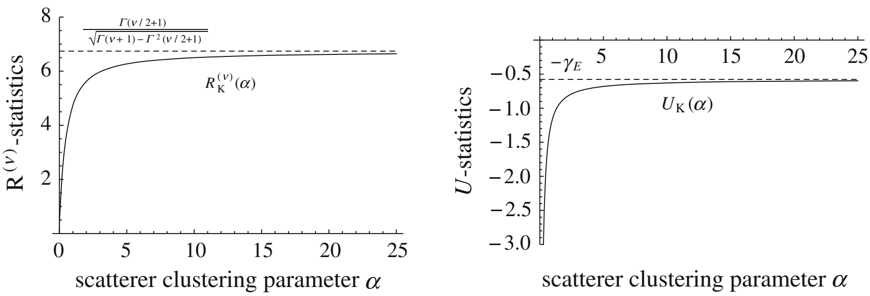

Fig. 10.7 Typical behavior of the $R^{(v)}$-statistics (*left image*) and of the U-statistics (*right image*) for the K-distribution. Here, $v = 1/4$

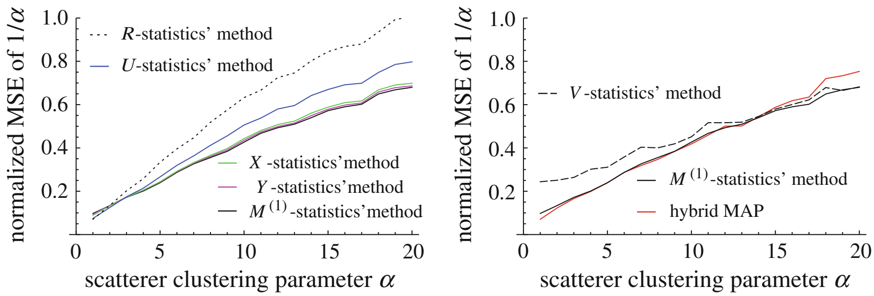

Fig. 10.8 *Left*: Comparison between the normalized MSE $\sqrt{E[(\hat{\beta}-\beta)^2]}/\beta$, where $\beta = 1/\alpha$, of the estimators based on the $M^{(1)}$-statistics (*black solid line*), the Y-statistics (*magenta solid line*), the X-statistics (*green solid line*), the U-statistics (*blue solid line*), and the R-statistics (*dotted line*), for the K-distribution. *Right*: Comparison between the normalized MSE of the estimators based on the $M^{(1)}$-statistics (*black solid line*), the V-statistics (*dashed line*), and the hybrid MAP (*red solid line*). The sample size is $N = 1,000$

normalized mean squared error (MSE) of the estimator $\hat{\beta}$ as $\sqrt{E[(\hat{\beta}-\beta)^2]}/\beta$. The resulting normalized MSE curves are presented in Fig. 10.8. As one can see, the estimators based on the $M^{(1)}$, Y, or X statistics and the hybrid MAP are practically equivalent and are better than the estimators based on the U or the R statistics.

10.4.5 Parameter Estimation Methods for the Homodyned K-Distribution

10.4.5.1 Expression of Fractional Order Moments of the Amplitude

Theorem 24 *Assume that $A = \sqrt{I}$ is distributed according to the homodyned K-distribution, with parameters $\varepsilon \geq 0$, $\sigma^2 > 0$ and $\alpha > 0$. Let $\gamma = \varepsilon^2/(2\sigma^2)$. Then,*

(a) (Prager et al. 2002) *if $\gamma \geq 0$, the $M^{(\nu)}$-statistics $E[A^\nu]/E[I]^{\nu/2}$ can be expressed in the following form*

$$M_{\mathrm{HK}}^{(\nu)}(\gamma,\alpha) = \frac{\Gamma(\nu/2+1)}{(\gamma+\alpha)^{\nu/2}} \int_0^\infty w^{\nu/2}\, {}_1F_1(-\nu/2, 1, -\frac{\gamma}{w}) \mathcal{G}(w \mid \alpha, 1)\, dw, \tag{10.44}$$

where ${}_pF_q$ denotes the hypergeometric series (here, $p = q = 1$).

(b) (Dutt and Greenleaf 1995) *if $\gamma = 0$*, the $M^{(\nu)}$-*statistics is equal to*

$$M_{\mathrm{HK}}^{(\nu)}(0,\alpha) = \Gamma(\nu/2+1)\frac{\Gamma(\nu/2+\alpha)}{\alpha^{\nu/2}\Gamma(\alpha)}. \tag{10.45}$$

(c) (Hruska and Oelze 2009) *if* $v/2 + \alpha$ *is not an integer and* $\gamma \geq 0$, *the* $M^{(v)}$-*statistics can be expressed as*

$$M_{\mathrm{HK}}^{(v)}(\gamma,\alpha) = \frac{\Gamma(v/2+1)}{(\gamma+\alpha)^{v/2}}\left\{\frac{\Gamma(v/2+\alpha)}{\Gamma(\alpha)}\,{}_1F_2(-v/2;1,1-v/2-\alpha;\gamma)\right.$$
$$\left.+\frac{\Gamma(v/2+1)\sin(\pi v/2)}{\Gamma^2(1+v/2+\alpha)\sin(\pi(v/2+\alpha))}\gamma^{v/2+\alpha}\,{}_1F_2(\alpha;1+v/2+\alpha,1+v/2+\alpha;\gamma)\right\}. \tag{10.46}$$

(d) (Jakeman and Tough 1987) *if* $v/2 > 2$ *is an integer and* $\gamma \geq 0$, *then the* $M^{(v)}$-*statistics is equal to*

$$M_{\mathrm{HK}}^{(v)}(\gamma,\alpha) = \frac{(v/2)!(v/2)!}{(\gamma+\alpha)^{v/2}\Gamma(\alpha)}\sum_{i=0}^{v/2}\frac{\Gamma(v/2-i+\alpha)}{i!i!(v/2-i)!}\gamma^i. \tag{10.47}$$

10.4.5.2 A Method Based on the Moments of the Intensity

A moments' method for the estimation of the homodyned K-distribution was proposed in Dutt and Greenleaf (1994). Namely, one solves the system of equations

$$E[I] = \overline{I}; \qquad E[I^2] = \overline{I^2}; \qquad E[I^3] = \overline{I^3} \tag{10.48}$$

in order to estimate $(\varepsilon, \sigma^2, \alpha)$, where $I = A^2$ is the intensity. In Prager et al. (2003), the three moments $E[I]$, $E[I^2]$, and $E[I^3]$ are expressed analytically as functions of $\tau^2 = \sigma^2\alpha$ (denoted σ^2 in that reference), $k = \varepsilon/(\sigma\sqrt{\alpha})$, and $\beta = 1/\alpha$, as follows

$$\begin{aligned}
E[I] &= \tau^2[k^2+2];\\
E[I^2] &= \tau^4[8(1+\beta)+8k^2+k^4];\\
E[I^3] &= \tau^6[48(1+3\beta+2\beta^2)+72k^2(1+\beta)+18k^4+k^6].
\end{aligned} \tag{10.49}$$

In Prager et al. (2003, Appendix C, p. 712), an algebraic method is presented to solve the system (10.49) for τ^2, k, and β, rejecting negative or imaginary values.

Observe that Eq. (10.48) is equivalent to the system of equations

$$\mu = \overline{I}; \qquad M_{\mathrm{HK}}^{(4)}(\gamma,\alpha) = \overline{I^2}/\overline{I}^2; \qquad M_{\mathrm{HK}}^{(6)}(\gamma,\alpha) = \overline{I^3}/\overline{I}^3, \tag{10.50}$$

where $\mu = \varepsilon^2 + 2\sigma^2\alpha = E[I]$. Moreover, the values of ε and σ^2 can be recovered from γ, α and μ with the change of variables

$$\varepsilon = \sqrt{\mu \frac{\gamma}{(\gamma + \alpha)}}; \qquad \sigma^2 = \mu \frac{1}{2(\gamma + \alpha)}. \tag{10.51}$$

10.4.5.3 A Method Based on the Moments of the Amplitude

In Dutt (1995, Sect. 9.2.2, p. 116), it was suggested to use the first three moments of the amplitude to estimate the homodyned K-distribution, namely, to solve the system of equations

$$E[A] = \overline{A}; \qquad E[A^2] = \overline{A^2}; \qquad E[A^3] = \overline{A^3}. \tag{10.52}$$

However, at that time, the authors could not find a closed form expression of the moments of the amplitude. So, approximate expressions were used instead. As noted in Dutt (1995, p. 117), the parameter estimation might break down for small values of α and large values of k, due to the weakness of the approximations.

Note that an explicit expression of an arbitrary moment of the amplitude was given in Hruska and Oelze (2009, Eq. (8), p. 2473). Thus, the estimation method based on the first three moments of the amplitude would likely need to be tested again with the exact expressions of those moments.

Observe that Eq. (10.52) is equivalent to the system of equations

$$\mu = \overline{I}; \qquad M_{\mathrm{HK}}^{(1)}(\gamma, \alpha) = \overline{A}/\overline{I}^{1/2}; \qquad M_{\mathrm{HK}}^{(3)}(\gamma, \alpha) = \overline{A^3}/\overline{I}^{3/2}. \tag{10.53}$$

10.4.5.4 Methods Based on the SNR of Fractional Order Moments of the Amplitude

In Dutt (1995, Sect. 9.2.4, p. 117), it was proposed to use the SNR R of the amplitude and of the intensity. It is reported in Dutt (1995, Sect. 9.5, p. 142) that the method based on SNRs gave better results than the three methods presented in Sects. 10.4.5.2, 10.4.5.3 and 10.4.5.5. But then, the exact expression of Eq. (10.46) was not used, so that this conclusion is not necessarily valid.

In Martin-Fernandez and Alberola-Lopez (2007), the authors suggested to use the statistics R for two distinct values of v (or more), using an exact expression of that statistics. In fact, the authors suggested the values 0.01, 0.03, 0.05, 0.075, 0.1, 0.25, 0.4, 0.5, 0.75, and 1. A solution is then found by inspection of the SNR level curves. Namely, for each value of the fractional order v, the statistics R is expressed analytically as a function of $k = \varepsilon/(\sigma\sqrt{\alpha})$, and α (denoted μ in that reference). One then considers the point (k, α) that is closest to all the corresponding SNR level curves, in the sense of the least mean squares (LMS). Thus, this method is an extension of the method based on the SNRs of Dutt (1995, Sect. 9.2.4).

10.4.5.5 A Method Based on the SNR and Skewness of the Amplitude

In Dutt (1995, Sect. 9.2.4, p. 117), it was proposed to use the SNR $R = \frac{E[A]}{(E[A^2]-E^2[A])^{1/2}}$ and the skewness $S = \frac{E[(A-E[A])^3]}{(E[A^2]-E^2[A])^{3/2}}$ of the amplitude for the estimation of the homodyned K-distribution. Again, that method should be tested with the exact expression of Eq. (10.46).

10.4.5.6 A Method Based on the SNR, Skewness and Kurtosis of Fractional Order Moments of the Amplitude

In Hruska and Oelze (2009), the authors suggested the use of the SNR R, the skewness S, and the kurtosis $K = \frac{E[(A^{\nu}-E[A^{\nu}])^4]}{(E[A^{2\nu}]-E^2[A^{\nu}])^2}$, for two values of ν, namely 0.72 and 0.88. These statistics were expressed analytically as a function of $k = \varepsilon/(\sigma\sqrt{\alpha})$, and α (denoted μ in that reference). One then considers the point (k, α) that is closest to the six corresponding SNR, skewness, and kurtosis level curves in the sense of the LMS. In order to do so, the (k, α)-space was sampled at the points of the form $(i \times 0.01, 10^{-3+j\times 0.01})$, with $0 \leq i, j \leq 500$. The two methods (Martin-Fernandez and Alberola-Lopez 2007; Hruska and Oelze 2009) were not compared in Hruska and Oelze (2009). However, the choice of the fractional orders 0.72 and 0.88 was validated empirically in Hruska and Oelze (2009) (as opposed to taking the numerous fractional orders $0.01, \ldots, 1$ in Martin-Fernandez and Alberola-Lopez (2007)).

10.4.5.7 Discussion

In this section, we present new results on the $M^{(\nu)}$-statistics and the MLE for the homodyned K-distribution.

Concerning Theorem 24, the case where $\nu/2 + \alpha$ is not integer (with no restriction on γ) is covered by part c), whereas part b) covers the case where $\nu/2 + \alpha$ is an integer, but with the restriction $\gamma = 0$. So, what about the case where $\gamma > 0$ and $\nu/2 + \alpha$ is an integer. The following result answers that question. However, in practice, one may use linear interpolation to approximate the $M^{(\nu)}$-statistics whenever $\nu/2 + \alpha$ is close to an integer (as done in Hruska and Oelze (2009)).

Theorem 25 *Assume that* $A = \sqrt{I}$ *is distributed according to the homodyned K-distribution, with parameters* $\varepsilon \geq 0$, $\sigma^2 > 0$ and $\alpha > 0$. *Let* $\gamma = \varepsilon^2/(2\sigma^2)$. *Then, if* $\gamma > 0$, *the* $M^{(\nu)}$*-statistics is expressed as*

$$M_{\mathrm{HK}}^{(v)}(\gamma,\alpha)=\frac{2}{(\gamma+\alpha)^{v/2}}\frac{\Gamma(v/2+1)}{\Gamma(\alpha)}\sum_{n=0}^{\infty}\frac{(1+v/2)_n}{n!n!}\sqrt{\gamma}^{v/2+\alpha+n}K_{v/2+\alpha-n}(2\sqrt{\gamma}). \tag{10.54}$$

where $(1+v/2)_n$ *denotes the rising factorial* $\Gamma(1+v/2+n)/\Gamma(1+v/2)$.

Theorems 9 and 21 on the behavior of the functions $M_{\mathrm{Ri}}^{(v)}(\kappa)$ and $M_{\mathrm{K}}^{(v)}(\alpha)$, respectively, can be extended to the following theorem.

Theorem 26 *Assume that* $A=\sqrt{I}$ *is distributed according to the homodyned K-distribution, with parameters* $\varepsilon\geq 0$, $\sigma^2>0$ *and* $\alpha>0$. *Let* $\gamma=\varepsilon^2/(2\sigma^2)$. *Then,*

(a) $\lim_{\gamma\to 0}M_{HK}^{(v)}(\gamma,\alpha)=M_K^{(v)}(\alpha)$ (*the function introduced in Theorem* 13).
(b) $\lim_{\gamma\to\infty}M_{\mathrm{HK}}^{(v)}(\gamma,\alpha)=1$.
(c) For any $\alpha>0$ *and* $\gamma>0$, *the function* $M_{HK}^{(v)}(\gamma,\alpha)$ *is increasing in the variable* γ, *if* $v<2$, *whereas it is decreasing in the variable* γ, *if* $v>2$.

See Fig. 10.9 for an illustration of Theorem 26. Theorem 26 implies that the equation $M_{\mathrm{HK}}^{(v)}(\gamma,\alpha)=M$, α being known, has at most one solution, and moreover, it gives sufficient and necessary conditions for a solution to exist, as expressed in the following Corollary.

Corollary 2 *Let* $M>0$ *be a real number (playing the role of* $M^{(v)}$*). There exists at most one non-negative solution* $\gamma=\gamma_M^{(v)}(\alpha)$ *to the equation* $M_{\mathrm{HK}}^{(v)}(\gamma,\alpha)=M$, α *being known.*

(a) *If the Rice conditions* (10.29) *are satisfied, then there exists a non-negative solution for any* $\alpha>0$.
(b) *If the K-distribution conditions* (10.43) *are satisfied, then there exists a non-negative solution if and only if* $\alpha\leq\alpha_0=(M_{\mathrm{K}}^{(v)})^{-1}(M)$.

So, given $\alpha>0$, $\overline{I}$ and $\overline{A}$, one can recover ε and σ^2 as follows. First of all, we take $\gamma=\gamma_M^{(v)}(\alpha)$, where $M=\overline{A}/(\overline{I})^{1/2}$. Then, one uses Eq. (10.51), with $\mu=\overline{I}$ and $\gamma=\gamma_M^{(v)}(\alpha)$. In this manner, well-defined functions can be obtained

$$\varepsilon(\alpha,\overline{I},\overline{A});\qquad \sigma^2(\alpha,\overline{I},\overline{A}), \tag{10.55}$$

We now discuss briefly the MLE for the homodyned K-distribution. The following result is useful for the computation of the partial derivatives of that distribution.

Lemma 4 *Let* $\varepsilon\geq 0$, $\sigma^2>0$ *and* $\alpha>0$.

(a) *The homodyned K-distribution* $P_{\mathrm{HK}}(A\,|\,\varepsilon,\sigma^2,\alpha)$ *can be expressed as*

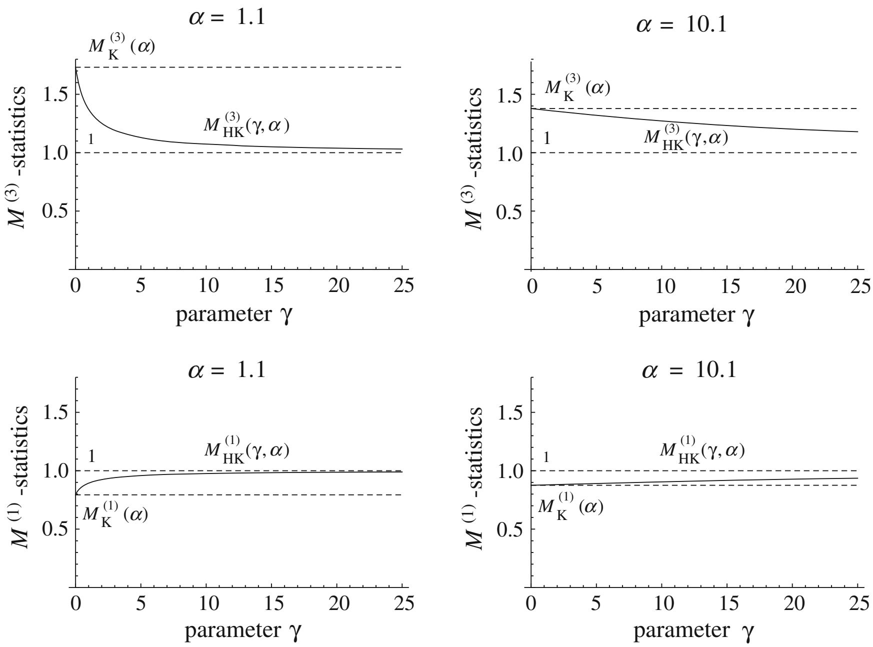

Fig. 10.9 Typical behavior of the $M^{(v)}$-statistics for the homodyned K-distribution, α being fixed, when $v < 2$ (*bottom row*) and $v > 2$ (*top row*). In fact, we took here $v = 1$ (*bottom row*) and $v = 3$ (*top row*), as well as $\alpha = 1.1$ (*left column*) and $\alpha = 10.1$ (*right column*)

$$\frac{1}{\pi}\int_0^{\pi} \frac{2A}{\sigma^2\Gamma(\alpha)}\left(\frac{X(\theta)}{2}\right)^{\alpha-1} K_{\alpha-1}\big(X(\theta)\big)d\ \theta, \tag{10.56}$$

where $X(\theta) = \sqrt{\frac{2}{\sigma^2}}\sqrt{A^2+\varepsilon^2-2A\varepsilon\cos\theta}$.

(b) *The partial derivative* $\frac{\partial}{\partial\varepsilon}P_{HK}(A\,|\,\varepsilon,\sigma^2,\alpha)$ *can be expressed as*

$$\frac{1}{\pi}\int_0^{\pi} \frac{2A}{\sigma^4\Gamma(\alpha)}(A\cos\theta-\varepsilon)\left(\frac{X(\theta)}{2}\right)^{\alpha-2} K_{\alpha-2}\big(X(\theta)\big)d\ \theta. \tag{10.57}$$

(c) *The partial derivative* $\frac{\partial}{\partial\sigma^2}P_{HK}(A\,|\,\varepsilon,\sigma^2,\alpha)$ *can be expressed as*

$$\begin{aligned}&-\frac{\alpha}{\sigma^2}\frac{1}{\pi}\int_0^{\pi} \frac{2A}{\sigma^2\Gamma(\alpha)}\left(\frac{X(\theta)}{2}\right)^{\alpha-1} K_{\alpha-1}\big(X(\theta)\big)d\ \theta\\ &+\frac{1}{\sigma^2}\frac{1}{\pi}\int_0^{\pi} \frac{2A}{\sigma^2\Gamma(\alpha)}\left(\frac{X(\theta)}{2}\right)^{\alpha} K_{\alpha}\big(X(\theta)\big)d\ \theta.\end{aligned} \tag{10.58}$$

(d) *The partial derivative* $\frac{\partial}{\partial\alpha}P_{HK}(A\,|\,\varepsilon,\sigma^2,\alpha)$ *can be expressed as*

$$
\begin{aligned}
&\frac{1}{\pi}\int_0^{\pi} \frac{2A}{\sigma^2\Gamma(\alpha)}\left(\frac{X(\theta)}{2}\right)^{\alpha-1} K_{\alpha-1}\Big(X(\theta)\Big) \\
&\qquad \times \left\{-\psi(\alpha) + \log\left(\frac{X(\theta)}{2}\right) + \frac{\frac{\partial}{\partial\alpha}K_{\alpha-1}\Big(X(\theta)\Big)}{K_{\alpha-1}\Big(X(\theta)\Big)}\right\} d\,\theta,
\end{aligned}
\tag{10.59}
$$

where ψ denotes the digamma function.

One could then extend Theorems 18 and 19 in the context of the homodyned K-distribution. In fact, we suspect that the MLE is not always well-defined for the homodyned K-distribution. Thus, one would have to consider a MAP estimator. Since such an estimator results in a time-consuming algorithm, we will not develop further that topic here.

10.4.6 Parameter Estimation Methods for the Nakagami Distribution

10.4.6.1 The MLE for the Nakagami Distribution

Since a Nakagami distribution on the amplitude A is equivalent to a gamma distribution on the intensity $I = A^2$, the estimation of the Nakagami distribution parameters amounts to the well-known estimation problem of the gamma distribution. In particular, the MLE is the unique solution to the equation

$$
\Omega = \overline{I}; \qquad \psi(m) - \log m = \overline{\log I} - \log \overline{I}, \tag{10.60}
$$

where ψ denotes the digamma function.

10.4.6.2 A Method Based on the First Two Moments of the Intensity

The most frequently used method for the parameter estimation of the Nakagami distribution is based on the first two moments of the intensity in the following form

$$
\Omega = \overline{I}; \qquad m = \frac{\overline{I}^2}{\overline{I^2} - \overline{I}^2}. \tag{10.61}
$$

Note that the term $\frac{\overline{I}^2}{\overline{I^2} - \overline{I}^2}$ is the square of the SNR of the intensity. This method is equivalent to the V-statistics' method (i.e., based on the $M^{(4)}$-statistics $\overline{I^2}/\overline{I}^2$).

10.4.6.3 Discussion

We first mention a relation between the MLE and the U-statistics for the Nakagami distribution. Then, new results on moments and log-moments based methods are presented. Finally, a comparison of these estimators on simulated data is reported.

Concerning the MLE of the Nakagami distribution, note that the term $\overline{\log I} - \log \bar{I}$ is the U-statistics. Thus, it is negative unless all terms I_i are identical. Also, the term $\psi(m) - \log m$ is the analytical expression of the U-statistics for the Nakagami distribution, as stated in the following result.

Theorem 27 *Let A be distributed according to the Nakagami distribution with parameters m and Ω. Then, the U-statistics* $E[\log I] - \log E[I]$ *is expressed as* $U_{\mathrm{Na}}(m) = \psi(m) - \log m$.

Thus, the MLE turns out to correspond to the U-statistics' method. But unlike the K-distribution, the equation $\psi(m) - \log m = U$ admits a solution for any $U < 0$. Indeed, the following result shows that a binary search can be used to compute the unique solution to that equation.

Theorem 28 (Destrempes et al. 2009). *The following properties hold*

a) $\lim_{m\to 0} \psi(m) - \log m = -\infty$;
b) $\lim_{m\to\infty} \psi(m) - \log m = 0$;
c) the function $\psi(m) - \log m$ *is strictly increasing on its domain* $(0, \infty)$.

Proof

a) We have the identity described in Abramowitz and Stegun (1972, p. 259, (6.3.21)) $\psi(m) = \log(m) - \frac{1}{2m} - 2\int_0^\infty \frac{t}{(t^2+m^2)(e^{2\pi t}-1)}$, for $m > 0$. This yields $-\log(m) + \psi(m) \leq -\frac{1}{2m}$, and hence $\lim_{m\to 0} - \log(m) + \psi(m) = -\infty$.
b) The same identity as above yields the computation $\lim_{m\to\infty} - \log(m) + \psi(m) = -2\lim_{m\to\infty} \int_0^\infty \frac{t}{(t^2+m^2)(e^{2\pi t}-1)}\, dt = 0$.
c) We have $\frac{\partial}{\partial m} U_{\mathrm{Na}}(m) = -\frac{1}{m} + \psi^{(1)}(m)$. But from Abramowitz and Stegun (1972, p. 260, (6.4.1)), we have $\psi^{(1)}(m) = \int_0^\infty \frac{te^{-mt}}{1-e^{-t}}\, dt$. Now, $\frac{te^{-\alpha t}}{1-e^{-t}} > e^{-mt}$, since $e^{-t} > 1 - t$, for $t > 0$. Therefore, $\psi^{(1)}(m) > \int_0^\infty e^{-mt}\, dt = \frac{1}{m}$, and we are done.

■

One can also show that the X-statistics is equal to $1/m$ for the Nakagami distribution.

Theorem 29 *Let A be distributed according to the Nakagami distribution with parameters m and Ω. Then, the X-statistic* $E[I \log I]/E[I] - \log E[I]$ *is expressed as* $X_{\mathrm{Na}}(m) = \frac{1}{m}$.

Thus, the shape parameter of the Nakagami distribution can be estimated directly with the equation $m = 1/X$, where $X = \overline{I \log I}/\bar{I} - \overline{\log I}$.

Finally, one can compute explicitly the $M^{(1)}$-statistics for the Nakagami distribution.

Theorem 30 *Let A be distributed according to the Nakagami distribution with parameters m and Ω. Then, the $M^{(1)}$-statistic $E[A]/\sqrt{E[I]}$ is expressed as $M^{(1)}_{\mathrm{Na}}(m) = \frac{\Gamma(1/2+m)}{\sqrt{m}\Gamma(m)}$.*

The equation $M^{(1)}_{\mathrm{Na}}(m) = M$ can be estimated with a binary search algorithm, for any $0 < M < 1$.

Theorem 31 *The following properties hold*

a) $\lim_{m\to 0} \frac{\Gamma(1/2+m)}{\sqrt{m}\Gamma(m)} = 0$;
b) $\lim_{m\to \infty} \frac{\Gamma(1/2+m)}{\sqrt{m}\Gamma(m)} = 1$;
c) the function $\frac{\Gamma(1/2+m)}{\sqrt{m}\Gamma(m)}$ is strictly increasing on its domain $(0, \infty)$.

Theorems 27, 29, 30 and 31 can be checked directly using the software Mathematica (Wolfram Research, Inc., Champaign, IL, USA, version 7.0).

In order to compare these four estimators, we considered the parameter m with values in the set $\{0.1, 0.2, \ldots, 2.0\}$. For each value of m, 1,000 datasets of $N =$ 1,000 elements each were simulated according to the corresponding Nakagami distribution. We could estimate the normalized MSE of the estimator $\hat{m}$ as $\sqrt{E[(\hat{m} - m)^2]}/m$. The resulting normalized MSE curves are presented in Fig. 10.10. As one can see, the estimators based on the MLE (i.e., the U-statistics, in this case) or the X-statistics are practically equivalent. They are better than the estimator based on the $M^{(1)}$-statistics, especially on the interval $m \in [0.0, 0.5]$. These three estimators are systematically better than the estimator based on the V-statistics.

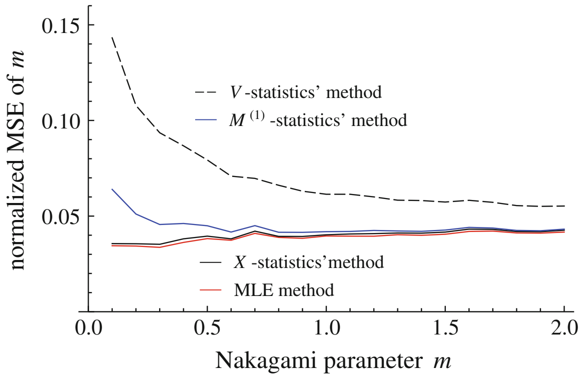

Fig. 10.10 Comparison between the normalized MSE $\sqrt{E[(\hat{m} - m)^2]}/m$ of the estimators based on the MLE (*red solid line*), the X-statistics (*black solid line*), the $M^{(1)}$-statistics (*blue solid line*), and the V-statistics (*dashed line*), for the Nakagami distribution. The sample size is $N = 1,000$

10.5 Conclusion

We conclude with the following issues.

1. It was argued that the homodyned K-distribution is a sound model for the first-order statistics of the echo envelope of the RF ultrasound signal, in the context where the backscattered echo signal received at the transducer of an ultrasound device is assumed to be the vector sum of the individual signals produced by the scatterers distributed in the medium (Wagner et al. 1983, 1987). The K-distribution is a special case where there is no coherent component (due to the absence of specular reflection). The Rice and the Rayleigh distributions are limit cases of the homodyned K-distribution or the K-distribution, respectively, corresponding to an infinite homogeneity of the diffuse scattering medium. The Nakagami is an approximation of the homodyned K-distribution. All these five distributions share two desirable properties: (1) the total signal power depends only on the coherent component in the case of a vanishing diffuse signal; and (2) the intensity SNR is infinite in that case. The other models presented in Jakeman and Tough (1987), Shankar (2000, 2003), Barakat (1986), Eltoft (2005), Raju and Srinivasan (2002), Agrawal and Karmeshu (2006) do not have these two properties. Thus, we recommend the homodyned K-distribution (or its related distributions, in special cases) as a model for the ultrasound echo envelope in that context, as was done in Dutt and Greenleaf (1994), Hruska and Oelze (2009) and Destrempes and Cloutier (2010).
2. It was shown that the methods based on the X-statistics and the mean intensity are practically as good as the MLE for the Rice and the Nakagami distributions, or the proposed hybrid MAP for the K-distribution. For the homodyned K-distribution, one may use a method based on the SNR, skewness and kurtosis of fractional orders of the amplitude (Hruska and Oelze 2009).
3. A homodyned K-distribution with parameters (k, α) in the range $[0, 2] \times [1, 20]$ can be approximated by a Nakagami distribution with KL distance less than 0.072 (but for much larger values of k, the KL distance might be much larger). However, although one may express the two parameters Ω and m of the Nakagami distribution in terms of the three parameters ε, σ^2 and α of the homodyned K-distribution in the form $\Omega = \varepsilon^2 + 2\sigma^2\alpha$ and $m = \frac{(\varepsilon^2+2\sigma^2\alpha)^2}{4\sigma^2\alpha(\varepsilon^2+\sigma^2(2+\alpha))}$, as follows from Destrempes and Cloutier (2010, Eq. (23) and Tables 3 and 4), the converse statement is not true. Thus, the Nakagami distribution gives less information on the statistical properties of the echo envelope than the homodyned K-distribution. In particular, one may not retrieve the coherent or diffuse signal power from the parameters of the Nakagami distribution. For this reason, we recommend the use of the homodyned K-distribution, rather than the Nakagami distribution, in the context of tissue characterization. On the other hand, the Nakagami distribution may be used in the context of image segmentation, since in that application, the main property is a good fit of the

distribution with the data. This was the point of view adopted in Destrempes et al. (2009, 2011) and Bouhlel and Sevestre-Ghalila (2009).

4. When the product of the wave number with the mean size of the scatterers is much smaller than the wavelength, and acoustic impedance of the scatterers is close to the impedance of the embedding medium, a high density of scatterers results in a packing organization that implies constructive and destructive wave interferences and a correlation between the individual signals produced by the scatterers (Hayley et al. 1967; Twersky 1975, 1978, 1987, 1988; Lucas and Twersky 1987; Berger et al. 1991). In such a case, the assumption made here on the randomness of the scatterer positions (or phase) might not be valid. The resulting first-order statistics might still be characterized with the proposed models, but the physical interpretation of the parameters should be done with caution in that case and should be further studied. See Wagner et al. (1987), Weng et al. (1990, 1992) and Narayanan et al. (1997) for further reading on that issue.
5. The distributions mentioned here concern the envelope of the RF signal. When a log-compression or other (nonlinear or linear) operators are applied to the envelope, the distribution of the gray levels no longer follows the distributions computed on the RF echo envelope. In the case of log-compression, the resulting distribution has been modeled in Dutt and Greenleaf (1996), assuming the K-distribution for the envelope. In Prager et al. (2003), a decompression algorithm is proposed, assuming the homodyned K-distribution for the envelope. As mentioned before, operators other than log-compression can be applied on the envelope. In Nillesen et al. (2008), a linear filter was applied to the RF data before computing the envelope. Five distributions were tested to fit the data: the Rayleigh distribution, the K-distribution, the Nakagami distribution, the inverse Gaussian distribution and the gamma distribution. The authors showed, based on empirical tests, that, overall, the gamma distribution best fits the data. See also Tao et al. (2002, 2006) and Shankar et al. (2003) for further reading on the gamma distribution model in ultrasound imaging. See also Keyes and Tucker (1999) for a comparison of the K-distribution with a few other models as well as Tsui et al. (2005, 2009c), Tsui and Wang (2004), Tsui and Chang (2007) for the effect of log-compression or transducer characteristics on the parameters of the Nakagami distribution. Here, we were concerned with the statistical distributions of the amplitude of the unfiltered envelope of the RF image, and therefore we did not study such distributions.
6. The parameters of the homodyned K-distribution reveal the scattering properties of the underlying tissue, but they are also instrumentation and depth dependent. In particular, the transducer center frequency, the point spread function (PSF) and the attenuation of the signal within the tissue play a role. A challenge consists in removing these dependencies. See Hruska (2009) for further reading on that matter.
7. The estimation problem is important, since the use of poor estimators might wash down the performance of a method, otherwise fine. However, the mere study of the bias and variance of an estimator is not sufficient, since it assumes

data distributed according to the distribution. Moreover, in the context of ultrasound imaging, various factors intervene, such as the presence of noise, the efficiency of the algorithm (in view of clinical applications, where the speed of execution of an algorithm is relevant). Thus, ideally, the study of an estimation method should include simulations of ultrasound data, as well as in vitro and in vivo experimental tests.

10.5.1 Perspective

In the future, it would be interesting to see a study of log-moments methods for the homodyned K-distribution. We believe that it would be instructive to establish relations between echo envelope statistics and spectral quantitative measures. See Oelze and O'Brien (2007) for an example of quantitative ultrasound assessment in the context of breast cancer that used the parameters of the homodyned K-distribution combined with an analysis of the normalized backscattered power spectrum. In the articles by Shankar et al. (1993), Molthen et al. (1993), Narayanan et al. (1994), Shankar (1995) and Molthen et al. (1995), an underlying physical model for the K-distribution was introduced. In the more recent article by Saha and Kolios (2011), the Nakagami distribution was estimated on simulated tissues based on a scattering model. A challenge consists in deepening the understanding of an underlying physical model for the homodyned K-distribution. Finally, it would be desirable to take into account the effect of instrumentation and attenuation on the echo envelope statistics. Thus, there remains several challenging problems in that area of QUS imaging, that we believe will turn out to be useful in a clinical context.

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Appendix: Proofs of the New Results

Proof of Theorem 8 From Theorem 6, there is exactly one critical point of $L(\varepsilon, \sigma^2)$ for which $\varepsilon > 0$, and it is the MLE (here, for L denotes L_{Ri}). Therefore (using Theorem 5), the function $f(\kappa)$ has exactly one positive root κ_* and it corresponds to the MLE. Moreover, one can check that $\kappa = 0$ is also a root of the function f. Namely, we have $\lim_{\kappa \to 0} f(\kappa) = -1 + \frac{1}{N}\sum_{i=1}^{N} y_i^2$, and by construction, $\frac{1}{N}\sum_{i=1}^{N} y_i^2 = 1$.

We have $\lim_{\kappa \to \infty} f(\kappa) = 2(-1 + \frac{1}{N}\sum_{i=1}^{N} y_i) = 2(-1 + \overline{\sqrt{I}}/\sqrt{\overline{I}})$. A direct application of Cauchy-Schwartz' inequality ensures that $\overline{\sqrt{I}}/\sqrt{\overline{I}} < 1$, so that

$\lim_{\kappa \to \infty} f(\kappa) < 0$. In view of the Intermediate Value Theorem for continuous functions, it follows that $f(\kappa) < 0$, if $\kappa > \kappa_*$.

Next, we want to show that $f(\kappa) > 0$ for $\kappa \in (0, \kappa_*)$. Since κ_* is the only positive root of f, and since $f(\kappa) < 0$ on (κ_*, ∞), it is enough to show that $\frac{\partial f}{\partial \kappa} < 0$ at κ_*; for then, $f(\kappa) > 0$ if $\kappa < \kappa_*$ is sufficiently near κ_*, and hence, $f(\kappa) > 0$ on $(0, \kappa_*)$ using the Intermediate Value Theorem.

First of all, we claim that $\frac{\partial f}{\partial \kappa} = \frac{1}{N} \frac{\partial^2 L}{\partial \kappa^2}$ at a critical point of $L(\varepsilon, \sigma^2) = \sum_{i=1}^{N} \log P_{\mathrm{Ri}}(A_i \mid \varepsilon, \sigma^2)$, whenever $\varepsilon > 0$ (i.e. $\kappa > 0$). Indeed, with the change of variable $\varepsilon = \sqrt{\frac{\mu\kappa}{(\kappa+1)}}$ and $\sigma^2 = \frac{\mu}{2(\kappa+1)}$, we obtain directly from Eq. (10.2)

$$\begin{aligned} \frac{1}{N} L(\mu, \kappa) = &\frac{1}{N} \sum_{i=1}^{N} \log A_i - \log \mu + \log 2 + \log(\kappa + 1) \\ &- \frac{(\kappa+1)}{\mu} \frac{1}{N} \sum_{i=1}^{N} A_i^2 - \kappa + \frac{1}{N} \sum_{i=1}^{N} \log I_0\Big(\frac{2A_i}{\sqrt{\mu}} \sqrt{\kappa(\kappa+1)}\Big). \end{aligned} \tag{10.62}$$

Next, the derivative of $\frac{1}{N} L(\mu, \kappa)$ with respect to κ is equal to

$$\frac{1}{(\kappa+1)} - \frac{1}{\mu} \frac{1}{N} \sum_{i=1}^{N} A_i^2 - 1 + \frac{(2\kappa+1)}{\sqrt{\kappa(\kappa+1)}} \frac{1}{N} \sum_{i=1}^{N} \frac{A_i}{\sqrt{\mu}} \frac{I_1\big(\frac{2A_i}{\sqrt{\mu}} \sqrt{\kappa(\kappa+1)}\big)}{I_0\big(\frac{2A_i}{\sqrt{\mu}} \sqrt{\kappa(\kappa+1)}\big)}. \tag{10.63}$$

But, from Talukdar and Lawing (1991), we have $\mu = \bar{I} = \frac{1}{N} \sum_{i=1}^{N} A_i^2$ at a critical point (ε, a^2) of L_{Ri}. Therefore, we obtain that $\frac{1}{N} \frac{\partial L}{\partial \kappa} = f(\kappa)$ at such a critical point (because $A_i / \sqrt{\mu}$ is then equal to $y_i = A_i / \sqrt{\bar{I}}$). Taking the partial derivative of Eq. (10.63) with respect to κ, we also see that $\frac{\partial f}{\partial \kappa} = \frac{1}{N} \frac{\partial^2 L}{\partial \kappa^2}$ at a critical point (ε, σ^2) of L.

Now, recall that if $u = u(x, y)$ and $v = v(x, y)$ is a change of variable, then

$$\begin{aligned} \frac{\partial^2 L}{\partial u \partial v} = &\frac{\partial L}{\partial x} \frac{\partial^2 x}{\partial u \partial v} + \frac{\partial L}{\partial y} \frac{\partial^2 y}{\partial u \partial v} \\ &+ \frac{\partial^2 L}{\partial x^2} \frac{\partial x}{\partial u} \frac{\partial x}{\partial v} + \frac{\partial^2 L}{\partial y \partial x} \frac{\partial y}{\partial u} \frac{\partial x}{\partial v} + \frac{\partial^2 L}{\partial x \partial y} \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} + \frac{\partial^2 L}{\partial y^2} \frac{\partial y}{\partial u} \frac{\partial y}{\partial v}; \\ \frac{\partial^2 L}{\partial^2 u} = &\frac{\partial L}{\partial x} \frac{\partial^2 x}{\partial u^2} + \frac{\partial L}{\partial y} \frac{\partial^2 y}{\partial u^2} \\ &+ \frac{\partial^2 L}{\partial x^2} \left(\frac{\partial x}{\partial u}\right)^2 + 2 \frac{\partial^2 L}{\partial x \partial y} \frac{\partial x}{\partial u} \frac{\partial y}{\partial u} + \frac{\partial^2 L}{\partial y^2} \left(\frac{\partial y}{\partial u}\right)^2. \end{aligned} \tag{10.64}$$

At this point, we find convenient to use the change of variable $\varepsilon^2 = \mu\kappa/(1+\kappa)$ and $\sigma^2 = \mu/(2(1+\kappa))$. We develop $\frac{\partial^2}{\partial \kappa^2} L = \left(G_{11} - G_{12} + \frac{1}{4} G_{22}\right) \frac{\mu^2}{(\kappa+1)^4}$ at a critical point of L, where $G_{11} = \frac{\partial^2 L}{\partial \varepsilon^2 \partial \varepsilon^2}$, $G_{12} = \frac{\partial^2 L}{\partial \varepsilon^2 \partial \sigma^2}$, and $G_{22} = \frac{\partial^2 L}{\partial \sigma^2 \partial \sigma^2}$ (we make use of the fact that $\frac{\partial L}{\partial \varepsilon^2} = 0 = \frac{\partial L}{\partial \sigma^2}$ at the critical point). Now, from Carrobi and Cati (2008, Appendix A, p. 686-687), we have $H_{11}H_{22} - H_{12}^2 > 0$ and $H_{11} < 0$ at the critical point of interest, where $H_{11} = \frac{\partial^2 L}{\partial\varepsilon\partial\varepsilon}$, $H_{12} = \frac{\partial^2 L}{\partial\varepsilon\partial\sigma^2}$, and $H_{22} = \frac{\partial^2 L}{\partial\sigma^2\partial\sigma^2}$ (σ^2 is viewed as a variable). From there, if one uses the change of variable $\varepsilon = \sqrt{\varepsilon^2}$ (and $\sigma^2 = \sigma^2$), one concludes that $G_{11}G_{22} - G_{12}^2 = \left(H_{11}H_{22} - H_{12}^2\right)\frac{1}{4\varepsilon^2} > 0$ and $G_{11} = H_{11}\frac{1}{4\varepsilon^2} < 0$, at that critical point. Thus, we obtain the upper bound $G_{22} < G_{12}^2/G_{11}$ (because $G_{11} < 0$), and therefore $\frac{\partial^2 L}{\partial\kappa^2} < \left(G_{11} - \quad G_{12} + \frac{1}{4}G_{12}^2/G_{11}\right)\frac{\mu^2}{(\kappa+1)^4}$. But this is equal to $\frac{1}{G_{11}}\frac{\mu^2}{(\kappa+1)^4}\left(G_{11} - \frac{1}{2}G_{12}\right)^2$, and it is non-positive since $G_{11} < 0$. Therefore, $\frac{\partial f}{\partial\kappa} = \frac{1}{N}\frac{\partial^2 L}{\partial\kappa^2} < 0$ at the point $\kappa = \kappa_*$ (with $\mu = \bar{I}$). This completes the proof of Theorem 8. ■

Proof of Theorem 9 (a) Setting $\kappa = 0$ in Theorem 7, we obtain directly $M_{\mathrm{Ri}}^{(v)}(0) = \Gamma(v/2+1)$.

(b) From Luke (1962, pp.7–8), we have the following asymptotic behavior

$$ {}_1F_1(a_1; b_1; z) \propto \frac{\Gamma(b_1)}{\Gamma(a_1)} z^{\chi} e^{z}\left(1 + \mathcal{O}(1/z)\right), \tag{10.65}$$

where $\chi = a_1 - b_1$, valid for $|\arg z| < \pi$ and $|z| \to \infty$. Therefore, we have

$$ {}_1F_1(v/2+1; 1; \kappa) \propto \frac{1}{\Gamma(v/2+1)} \kappa^{v/2} e^{\kappa}\left(1 + \mathcal{O}(1/\kappa)\right). \tag{10.66}$$

We conclude that $\lim_{\kappa\to\infty} M_{\mathrm{Ri}}^{(v)}(\kappa) = \lim_{\kappa\to\infty} \frac{\Gamma(v/2+1)e^{-\kappa}}{(\kappa+1)^{v/2}} \times \frac{1}{\Gamma(v/2+1)}\kappa^{v/2}e^{\kappa} = 1$.

(c) From the definition $M_{\mathrm{Ri}}^{(v)}(\kappa) = \Gamma(v/2+1)\frac{{}_1F_1(1+v/2,1,\kappa)}{e^{\kappa}(\kappa+1)^{v/2}}$, we obtain after algebraic simplifications

$$ \frac{d}{d\kappa} M_{\mathrm{Ri}}^{(v)} = \Gamma(v/2+1)\frac{\frac{d}{d\kappa}{}_1F_1(1+v/2,1,\kappa) - {}_1F_1(1+v/2,1,\kappa)\left(1 + \frac{v}{2}(\kappa+1)^{-1}\right)}{e^{\kappa}(\kappa+1)^{v/2}}. \tag{10.67}$$

Now, from Gradshteyn and Ryshik (1994, 9.213, p.1086) and Gradshteyn and Ryshik (1994, 9.212(3), p.1086), we have $\frac{d}{d\kappa}{}_1F_1(1+v/2,1,\kappa) = (1+v/2)\,{}_1F_1(2+v/2,2,\kappa) = \frac{v}{2}\,{}_1F_1(1+v/2,2,\kappa) + {}_1F_1(1+v/2,1,\kappa)$. So, omitting the positive factor $\Gamma(v/2+1)e^{-\kappa}(\kappa+1)^{-v/2}$, we obtain

$$ \frac{v}{2}\,{}_1F_1(1+v/2,2,\kappa) - \frac{v}{2}\,{}_1F_1(1+v/2,1,\kappa)(\kappa+1)^{-1}. \tag{10.68}$$

Multiplying by $(\kappa+1)$ and dividing by $v/2$ (both are positive numbers), we obtain

$$ {}_1F_1(1+v/2,2,\kappa)(\kappa+1) - {}_1F_1(1+v/2,1,\kappa). \tag{10.69}$$

Using (Gradshteyn and Ryshik 1994, 9.212(2), p.1086), we have $\kappa\, {}_1F_1(1+v/2,2,\kappa) - {}_1F_1(1+v/2,1,\kappa) = -\,{}_1F_1(v/2,1,\kappa)$. Therefore, we finally obtain (up to a positive constant)

$$ {}_1F_1(1+v/2,2,\kappa) - {}_1F_1(v/2,1,\kappa). \tag{10.70}$$

Now, by definition, the hypergeometric function ${}_1F_1(a,b,z)$ is equal to $\sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n}\frac{z^n}{n!}$, where $(a)_n = a(a+1)...(a+n-1)$ is the rising factorial. If $v/2<1$, then $\frac{(1+v/2)_n}{(2)_n} > \frac{(v/2)_n}{(1)_n}$ and hence ${}_1F_1(1+v/2,2,\gamma) - {}_1F_1(v/2,1,\gamma) > 0$, On the other hand, if $v/2>1$, then $\frac{(1+v/2)_n}{(2)_n} < \frac{(v/2)_n}{(1)_n}$ and hence ${}_1F_1(1+v/2,2,\kappa) - {}_1F_1(v/2,1,\kappa) < 0$. This completes the proof of the theorem. ■

Proof of Theorem 10 (a) First of all, using the change of variable $I=A^2$, one computes

$$\int_0^{\infty} \log A^2 P_{\mathrm{Ri}}(A \mid \varepsilon,\sigma^2)\, dA = \int_0^{\infty} \log I \frac{1}{2\sigma^2} I_0\Big(\frac{\varepsilon}{\sigma^2}\sqrt{I}\Big) e^{-\varepsilon^2/(2\sigma^2)} e^{-I/(2\sigma^2)}\, dI, \tag{10.71}$$

which is a Laplace transform equal to $\Gamma(0,\frac{\varepsilon^2}{2\sigma^2}) + \log \varepsilon^2$, where $\Gamma(0,x)$ is the incomplete gamma function $\int_x^{\infty}\frac{e^{-t}}{t}\,dt$. Then, after subtraction by the term $\log(\varepsilon^2+2\sigma^2)$, one obtains $\Gamma(0,\frac{\varepsilon^2}{2\sigma^2}) + \log(\frac{\varepsilon^2}{\varepsilon^2+2\sigma^2})$, which is equal to $\Gamma(0,\kappa) + \log(\frac{\kappa}{\kappa+1})$ (where $\kappa = \varepsilon^2/(2\sigma^2)$).

(b) Again, using the change of variable $I=A^2$, we compute

$$\begin{aligned}&\int_0^{\infty} A^2 \log A^2 P_{\mathrm{Ri}}(A \mid \varepsilon,\sigma^2)\, dA \\ &= \int_0^{\infty} I \,\log I \frac{1}{2\sigma^2} I_0\Big(\frac{\varepsilon}{\sigma^2}\sqrt{I}\Big) e^{-\varepsilon^2/(2\sigma^2)} e^{-I/(2\sigma^2)}\, dI.\end{aligned} \tag{10.72}$$

This Laplace transform is equal to $4\sigma^2 - 2e^{-\frac{\varepsilon^2}{2\sigma^2}}\sigma^2 + (\varepsilon^2+2\sigma^2)\big(\Gamma(0,\frac{\varepsilon^2}{2\sigma^2}) + \log\varepsilon^2\big)$. Dividing by $\varepsilon^2+2\sigma^2$ and subtracting $E[\log I] = \Gamma(0,\frac{\varepsilon^2}{2\sigma^2}) + \log\varepsilon^2$ (from the proof of part a)), one obtains $(4\sigma^2 - 2e^{-\frac{\varepsilon^2}{2\sigma^2}}\sigma^2) \div (\varepsilon^2+2\sigma^2)$, which is equal to $\frac{1}{\kappa+1}(2-e^{-\kappa})$, after algebraic simplifications. ■

Proof of Theorem 11 (a) From Abramowitz and Stegun (1972, (6.5.15), p. 262), we have $\Gamma(0,\kappa) = E_1(\kappa)$ (the exponential integral). Moreover, from Abramowitz and Stegun (1972, (5.1.11), p. 229), $E_1(\kappa) = -\gamma_E - \log\kappa + \sum_{n=1}^{\infty}\frac{(-1)^n\kappa^n}{nn!}$. We conclude that $U_{\mathrm{Ri}}(\kappa) = -\gamma_E - \log(1+\kappa) + \sum_{n=1}^{\infty}\frac{(-1)^n\kappa^n}{nn!}$. Henceforth, $\lim_{\kappa\to 0} U_{\mathrm{Ri}}(\kappa) = -\gamma_E$.

(b) Since $\Gamma(0,\kappa) = \int_{\kappa}^{\infty}\frac{e^{-t}}{t}\,dt$, it follows that $\lim_{\kappa\to\infty}\Gamma(0,\kappa) = 0$. Moreover, $\lim_{\kappa\to\infty}\log\frac{\kappa}{\kappa+1} = 0$.

(c) We compute $\frac{d}{d\kappa}U_{\mathrm{Ri}}(\kappa) = -\frac{e^{-\kappa}}{\kappa} + \frac{1}{\kappa} - \frac{1}{\kappa+1}$. This is positive because $e^{\kappa} > 1 + \kappa$, for $\kappa > 0$. ■

Proof of Theorem 12 Parts (a) and (b) follow from basic Calculus.

(c) We compute $\frac{d}{d\kappa}X_{\mathrm{Ri}}(\kappa) = \frac{e^{-\kappa}(\kappa+1)-(2-e^{-\kappa})}{(\kappa+1)^2}$. Ignoring the positive factor $1/(\kappa+1)^2$, we obtain $e^{-\kappa}(2+\kappa) - 2$. This is negative since $e^{\kappa} > 1 + \kappa/2$, for $\kappa > 0$. ■

Lemma 5 *Let* $\alpha > 0$ *be fixed. Denote any root of* $\frac{\partial}{\partial\sigma^2}L(\sigma^2, \alpha)$ by $\sigma^2(\alpha, \tilde{A})$, *where* $\tilde{A} = \{A_1, A_2, ..., A_N\}$.

(a) *If* $0 < \alpha \leq 1/2$, *then* $\sigma^2(\alpha, \tilde{A}) \geq \frac{\alpha+\alpha^2+\sqrt{2\alpha^3+\alpha^4}}{\alpha^2}\left(\overline{A}\right)^2$.
(b) *If* $1/2 < \alpha \leq 3$, *then* $\sigma^2(\alpha, \tilde{A}) \geq \frac{1}{2\alpha^2}\left(\overline{A}\right)^2$.
(c) *If* $\alpha > 3$, *then* $\sigma^2(\alpha, \tilde{A}) \geq \frac{2\alpha-3+\sqrt{4\alpha-7}}{4(\alpha-2)^2}\left(\overline{A}\right)^2$.
(d) *If* $0 < \alpha \leq 1/2$, *then* $\sigma^2(\alpha, \tilde{A}) \leq \frac{1}{2\alpha^2}\left(\overline{A}\right)^2$.
(e) *If* $1/2 < \alpha \leq 3/2$, *then* $\sigma^2(\alpha, \tilde{A}) \leq \frac{1}{2(\alpha/2+1/4)^2}\left(\overline{A}\right)^2$.
(f) *If* $3/2 < \alpha \leq 3$, *then* $\sigma^2(\alpha, \tilde{A}) \leq \frac{1}{2}\left(\overline{A}\right)^2$.
(g) *If* $\alpha > 3$, *then* $\sigma^2(\alpha, \tilde{A}) \leq \frac{1}{2(\alpha-2)}\overline{A^2}$.
(h) *The function* $\frac{\partial}{\partial\sigma^2}L(\sigma^2, \alpha)$ *is positive at the lower bounds mentioned in parts (a) to (c), whereas it is negative at the upper bounds of parts (d) to (g).*

Proof We compute

$$\sigma^2 \frac{\frac{\partial}{\partial\sigma^2}P_{\mathrm{K}}(A \mid \sigma^2, \alpha)}{P_{\mathrm{K}}(A \mid \sigma^2, \alpha)} = -\alpha + \left(\frac{1}{\sqrt{2\sigma^2}}A\right)\frac{K_\alpha\left(\sqrt{\frac{2}{\sigma^2}}A\right)}{K_{\alpha-1}\left(\sqrt{\frac{2}{\sigma^2}}A\right)}. \tag{10.73}$$

Part a). If $0 < \alpha \leq 1/2$, then $K_{\alpha-1}(x) = K_{1-\alpha}(x) < K_1(x)$ and $K_\alpha(x) > K_0(x)$ for any $x > 0$. Also, the inequality $K_0(x)/K_1(x) > 1 - \frac{1}{(x+1)}$ holds for any $x > 0$. Therefore, from Eq. (10.73), we obtain $\sigma^2 \frac{\frac{\partial}{\partial\sigma^2}P_{\mathrm{HK}}(A \mid \sigma^2, \alpha)}{P_{\mathrm{HK}}(A \mid \sigma^2, \alpha)} > -\alpha + \frac{1}{2}f(\sqrt{\frac{2}{\sigma^2}}A)$, where $f(x) = x(1 - \frac{1}{(x+1)})$. Thus, we obtain that

$$\sigma^2 \frac{\partial}{\partial\sigma^2}L(\sigma^2, \alpha) > -N\alpha + \frac{1}{2}\sum_{i=1}^{N} f(\sqrt{\frac{2}{\sigma^2}}A_i). \tag{10.74}$$

Here, L denotes L_{K}. Now, the function $f(x)$ is convex. Therefore, from Jensen's inequality (Jensen 1906), we conclude that

$$\sigma^2 \frac{\partial}{\partial\sigma^2}L(\sigma^2, \alpha) > -N\alpha + \frac{N}{2}f(\sqrt{\frac{2}{\sigma^2}}\overline{A}). \tag{10.75}$$

But the right-hand side of Eq. (10.75) is positive if $\sigma^2 < \frac{\alpha+\alpha^2+\sqrt{2\alpha^3+\alpha^4}}{\alpha^2}\left(\overline{A}\right)^2$. This proves part a).

Part b). If $\alpha > 1/2$, then $K_\alpha(x) > K_{\alpha-1}(x)$ for any $x > 0$. Therefore, from Eq. (10.73), we obtain $\sigma^2 \frac{\frac{\partial}{\partial\sigma^2}P_K(A\mid\sigma^2,\alpha)}{P_K(A\mid\sigma^2,\alpha)} > -\alpha + \frac{1}{\sqrt{2\sigma^2}}A$. Thus, we conclude that

$$\sigma^2 \frac{\partial}{\partial\sigma^2} L(\sigma^2,\alpha) > -N\alpha + \frac{1}{\sqrt{2\sigma^2}} \sum_{i=1}^N A_i. \tag{10.76}$$

But the right-hand side of Eq. (10.76) is positive if $\sigma^2 < \frac{1}{2\alpha^2}\left(\overline{A}\right)^2$. This proves part b).

Part c). If $\alpha > 3$, then $\frac{x}{2}\frac{K_\alpha(x)}{K_{\alpha-1}(x)} > \frac{x}{2} \times \left(\frac{2(\alpha-1)}{x} + \frac{1}{\frac{2(\alpha-2)}{x}+1}\right)$. Thus, $\sigma^2 \frac{\frac{\partial}{\partial\sigma^2}P_K(A\mid\sigma^2,\alpha)}{P_K(A\mid\sigma^2,\alpha)}$ has lower bound $-1+f\left(\frac{A}{\sqrt{2\sigma^2}}\right)$, where $f(x) = \left(\frac{(\alpha-2)}{x^2} + \frac{1}{x}\right)^{-1}$. Thus, we conclude that

$$\sigma^2 \frac{\partial}{\partial\sigma^2} L(\sigma^2,\alpha) > -N + \sum_{i=1}^N \frac{1}{\frac{2\sigma^2(\alpha-2)}{A_i^2} + \frac{\sqrt{2\sigma^2}}{A_i}}. \tag{10.77}$$

From Jensen's inequality, we then obtain

$$\sigma^2 \frac{\partial}{\partial\sigma^2} L(\sigma^2,\alpha) > -N + N \frac{1}{\frac{2\sigma^2(\alpha-2)}{\overline{A}^2} + \frac{\sqrt{2\sigma^2}}{\overline{A}}}, \tag{10.78}$$

because the function $f(x)$ above is convex. But the right-hand side of Eq. (10.78) is positive if $\sigma^2 < \frac{2\alpha-3+\sqrt{4\alpha-7}}{4(\alpha-2)^2}\left(\overline{A}\right)^2$. This proves part c).

Part d). If $0<\alpha\le 1/2$, then $K_\alpha(x) < K_{\alpha-1}(x)$ for any $x > 0$. Therefore, from Eq. (10.73), we obtain $\sigma^2 \frac{\frac{\partial}{\partial\sigma^2}P_K(A\mid\sigma^2,\alpha)}{P_K(A\mid\sigma^2,\alpha)} < -\alpha + \frac{1}{\sqrt{2\sigma^2}}A$. This yields the inequality

$$\sigma^2 \frac{\partial}{\partial\sigma^2} L(\sigma^2,\alpha) < -N\alpha + \frac{1}{\sqrt{2\sigma^2}} \sum_{i=1}^N A_i. \tag{10.79}$$

But the right-hand side of Eq. (10.79) is negative if $\sigma^2 > \frac{1}{2\alpha^2}\left(\overline{A}\right)^2$. This proves part d).

Part e). If $1/2<\alpha\le 3/2$, then $\frac{K_\alpha(x)}{K_{\alpha-1}(x)} < 1 + \frac{(\alpha-1/2)}{x}$ for any $x > 0$. Therefore, we have $\sigma^2 \frac{\frac{\partial}{\partial\sigma^2}P_K(A\mid\sigma^2,\alpha)}{P_K(A\mid\sigma^2,\alpha)} < -\frac{\alpha}{2} - \frac{1}{4} + \frac{1}{\sqrt{2\sigma^2}}A$. It follows that

$$\sigma^2 \frac{\partial}{\partial\sigma^2} L(\sigma^2,\alpha) < -N(\alpha/2+1/4) + \frac{1}{\sqrt{2\sigma^2}} \sum_{i=1}^N A_i. \tag{10.80}$$

But the right-hand side of Eq. (10.80) is negative if $\sigma^2 > \frac{1}{2(\alpha/2+1/4)^2}\left(\overline{A}\right)^2$. This proves part e).

Part f). If $3/2<\alpha\leq 3$, then $\frac{K_\alpha(x)}{K_{\alpha-1}(x)}<1+\frac{2(\alpha-1)}{x}$ for any $x>0$. Thus, we obtain $\sigma^2\frac{\frac{\partial}{\partial\sigma^2}P_{\mathrm{K}}(A\,|\,\sigma^2,\alpha)}{P_{\mathrm{K}}(A\,|\,\sigma^2,\alpha)}<-\alpha+(\alpha-1)+\frac{1}{\sqrt{2\sigma^2}}A$. From there, we conclude that

$$\sigma^2\frac{\partial}{\partial\sigma^2}L(\sigma^2,\alpha)<-N+\frac{1}{\sqrt{2\sigma^2}}\sum_{i=1}^{N}A_i. \tag{10.81}$$

But the right-hand side of Eq. (10.81) is negative if $\sigma^2>\frac{1}{2}\left(\overline{A}\right)^2$. Hence, part f) of the Theorem.

Part g). If $3<\alpha$, then $\frac{x}{2}\frac{K_\alpha(x)}{K_{\alpha-1}(x)}<(\alpha-1)+\frac{x^2}{4(\alpha-2)}$ for any $x>0$. Therefore, we obtain

$$\sigma^2\frac{\partial}{\partial\sigma^2}L(\sigma^2,\alpha)<-N+\frac{1}{2\sigma^2(\alpha-2)}\sum_{i=1}^{N}A_i^2. \tag{10.82}$$

But the right-hand side of Eq. (10.82) is negative if $\sigma^2>\frac{1}{2(\alpha-2)}\overline{A^2}$. Hence, part g) of the Theorem.

Finally, part h) follows from the proof of parts a) to g). ■

Proof of Theorem 18 From Lemma 5, for any $\alpha>0$, there exist two values $0<\sigma_1^2<\sigma_2^2$ for which $\frac{\partial}{\partial\sigma^2}L(\sigma_1^2,\alpha)>0$ and $\frac{\partial}{\partial\sigma^2}L(\sigma_2^2,\alpha)<0$, where L denotes L_{K}. Thus, by the Intermediate Value Theorem, there exists $\sigma^2=\sigma^2(\alpha,\tilde{A})$ such that $\frac{\partial}{\partial\sigma^2}L(\sigma^2,\alpha)=0$. ■

Proof of Theorem 19 Part a). Let $0<\alpha<1/2$. In Eq. (10.32), the term $-\psi(\alpha)+\log\left(\frac{x}{2}\right)-\frac{\frac{\partial}{\partial\alpha}K_{1-\alpha}(x)}{K_{1-\alpha}(x)}$ is an increasing function of $x>0$. Also, from Lemma 5 part d), we have $\sigma^2(\alpha,\tilde{A})\leq\frac{1}{2\alpha^2}\left(\overline{A}\right)^2$. Therefore, we obtain $\sqrt{\frac{2}{\sigma^2}}A_i\geq 2\alpha\frac{A_i}{\overline{A}}$. It follows that $LB(\alpha)=-\psi(\alpha)+\log\left(\alpha\frac{A_i}{\overline{A}}\right)-\frac{\frac{\partial}{\partial\alpha}K_{1-\alpha}\left(2\alpha\frac{A_i}{\overline{A}}\right)}{K_{1-\alpha}\left(2\alpha\frac{A_i}{\overline{A}}\right)}$ is a lower bound for that term. Now, from Abramowitz and Stegun (1972, Eq. (9.6.45), p. 377), we have $\frac{\frac{\partial}{\partial\alpha}K_{1-\alpha}(x)}{K_{1-\alpha}(x)}\approx\frac{\frac{\partial}{\partial\alpha}K_1(x)}{K_1(x)}=\frac{K_0(x)}{xK_1(x)}$ as $\alpha\to 0$. Moreover, from Abramowitz and Stegun (1972, Eqs. (9.6.8) and (9.6.9), p. 375), we have $\frac{K_0(x)}{xK_1(x)}\sim-\log x$ for small values of $x>0$. But $x=2\alpha\frac{A_i}{\overline{A}}$ has small values for $\alpha\to 0$. Thus, we obtain $\lim_{\alpha\to 0}\alpha LB(\alpha)=\lim_{\alpha\to 0}\alpha\left\{-\psi(\alpha)+\log\left(\alpha\frac{A_i}{\overline{A}}\right)+\log\left(2\alpha\frac{A_i}{\overline{A}}\right)\right\}=1$. This proves part a).

Part b). First of all, we observe that there exist constants $0<C_1<C_2$, such that $\frac{1}{C_2}\leq\liminf_{\alpha\to\infty}\frac{\sigma^2(\alpha,\tilde{A})}{1/\alpha}\leq\limsup_{\alpha\to\infty}\frac{\sigma^2(\alpha,\tilde{A})}{1/\alpha}\leq\frac{1}{C_1}$. The first inequality follows from Lemma 5 part c), whereas the third inequality follows from Lemma 5 part g).

Let L denote L_{K}. Since by definition $\frac{\partial}{\partial\sigma^2}L(\sigma^2,\alpha)|_{\sigma^2(\alpha,\tilde{A})}=0$, we might as well consider the expression $\alpha\frac{\partial}{\partial\alpha}L(\sigma^2,\alpha)-\sigma^2\frac{\partial}{\partial\sigma^2}L(\sigma^2,\alpha)$. From Eqs. (10.32) and (10.33), each term of that expression is equal to

$$\begin{aligned}&-\alpha\psi(\alpha)+\alpha\log\Big(\frac{1}{\sqrt{2\sigma^2}}A_i\Big)+\alpha\frac{\frac{\partial}{\partial\alpha}K_{\alpha-1}\Big(\sqrt{\frac{2}{\sigma^2}}A_i\Big)}{K_{\alpha-1}\Big(\sqrt{\frac{2}{\sigma^2}}A_i\Big)}\\&+\alpha-\Big(\frac{1}{\sqrt{2\sigma^2}}A_i\Big)\frac{K_\alpha\big(\sqrt{\frac{2}{\sigma^2}}A_i\big)}{K_{\alpha-1}\big(\sqrt{\frac{2}{\sigma^2}}A_i\big)}.\end{aligned}\tag{10.83}$$

From Abraham and Lyons (2002, Eq. (46)), we have $\frac{\frac{\partial}{\partial\alpha}K_{\alpha-1}(x)}{K_{\alpha-1}(x)}\sim\psi(\alpha-1)-\log(x/2)+\frac{x^2}{4\alpha^2}$ for large values of α. Also, from Abraham and Lyons (2002, Eq. (45)), we have $K_\alpha(x)\sim\frac{\Gamma(\alpha)}{2(x/2)^\alpha}\Big(1-\frac{(x/2)^2}{(\alpha-1)(\alpha-2)}\Big)^{\alpha-2}$ for large values of α. Therefore, taking $\sigma^2=1/(C\alpha)$, we obtain the asymptotic expression

$$-\alpha\psi(\alpha)+\alpha\psi(\alpha-1)+\frac{CA_i^2}{2}+\alpha-(\alpha-1)\frac{\Big(1-\frac{C\alpha A_i^2}{2(\alpha-1)(\alpha-2)}\Big)^{\alpha-2}}{\Big(1-\frac{C\alpha A_i^2}{2(\alpha-2)(\alpha-3)}\Big)^{\alpha-3}}.\tag{10.84}$$

Finally, Eq. (10.84) tends to 0 as α tends to infinity. This proves part b). ■

Proof of Theorem 20 Part a). Let $0<\alpha<1$. We consider again Eq. (10.83). Using the asymptotic forms (for small values of x and of α) $\frac{\frac{\partial}{\partial\alpha}K_{1-\alpha}(x)}{K_{1-\alpha}(x)}\approx\frac{K_0(x)}{xK_1(x)}$, $\frac{K_0(x)}{xK_1(x)}\sim-\log x$ and $\big(\frac{x}{2}\big)\frac{K_0(x)}{K_1(x)}\sim-\frac{x^2}{2}\log x$, and setting $x=\sqrt{\frac{2}{\sigma^2}}A_i$ with $\sigma^2=1/(C\alpha)$ and $C=2/\bar{I}$, we obtain the asymptotic expression

$$-\alpha\psi(\alpha)+\alpha\log\big(\frac{1}{2}\sqrt{\alpha}\sqrt{C}A_i\big)+\alpha\log\big(\sqrt{\alpha}\sqrt{C}A_i\big)+\alpha+\alpha^2CA_i^2\log\big(\sqrt{\alpha}\sqrt{C}A_i\big).\tag{10.85}$$

Part a) then follows by taking the limit of Eq. (10.85) as $\alpha\to 0$.

Part b). Taking $\sigma^2=1/(C\alpha)$, where $C=2/\bar{I}$, into Eq. (10.84), we obtain the limit 0 as $\to\infty$. This proves part b). ■

Proof of Theorem 21 a) At $\alpha=0$, we have $\Gamma(\alpha+v/2)=\Gamma(v/2)$. Also, $\Gamma(\alpha)$ has a simple pole with residue 1 at $\alpha=0$. Therefore, $\frac{\Gamma(\alpha+1/2)}{\alpha^{v/2}\Gamma(\alpha)}\sim\Gamma(v/2)\alpha^{1-v/2}$ at $\alpha\approx 0$, which shows part a).

b) Using Sterling's formula, we have $\frac{\Gamma(\alpha+v/2)}{\alpha^{v/2}\Gamma(\alpha)}\sim\frac{e^{-\alpha-v/2}(\alpha+v/2)^{\alpha+v/2-1/2}}{\alpha^{v/2}e^{-\alpha}\alpha^{\alpha-1/2}}$, which is equal to $e^{-v/2}\Big(1+\frac{v/2}{\alpha}\Big)^\alpha\Big(1+\frac{v/2}{\alpha}\Big)^{v/2-1/2}$. Therefore, $\lim_{\alpha\to\infty}\frac{\Gamma(\alpha+v/2)}{\alpha^{v/2}\Gamma(\alpha)}=1$, and we are done.

c) Using the logarithmic derivative, we have $\frac{d}{d\alpha}M_{\mathrm{K}}^{(v)}(\alpha) = M_{\mathrm{K}}^{(v)}(\alpha)\left(\psi(\alpha+v/2) - \psi(\alpha) - \frac{v}{2\alpha}\right)$. Now, we have $M_{\mathrm{K}}^{(v)}(\alpha) > 0$. Also, $\frac{1}{\alpha} = \psi(\alpha+1) - \psi(\alpha)$, and hence $\psi(\alpha+v/2) - \psi(\alpha) - \frac{v}{2\alpha} = \psi(\alpha+v/2) - \left(\frac{v}{2}\psi(\alpha+1) + (1-\frac{v}{2})\psi(\alpha)\right)$. Since the function ψ is convex, we conclude that $\psi(\alpha+v/2) - \left(\frac{v}{2}\psi(\alpha+1) + (1-\frac{v}{2})\psi(\alpha)\right) > 0$, if $v/2 < 1$, whereas it is negative if $v/2 > 1$. ■

Proof of Theorem 22 a) We consider the function $f(\alpha) = \frac{\Gamma(v+1)\Gamma(v+\alpha)\Gamma(\alpha)}{\Gamma^2(v/2+1)\Gamma^2(v/2+\alpha)}$, noting that $R_{\mathrm{K}}^{(v)} = \left(f(\alpha)-1\right)^{-1/2}$. Now, as $\alpha \to 0$, we have $\Gamma(\alpha) \to \infty$, whereas $\frac{\Gamma(v+1)\Gamma(v+\alpha)}{\Gamma^2(v/2+1)\Gamma^2(v/2+\alpha)} \to \frac{\Gamma(v+1)\Gamma(v)}{\Gamma^2(v/2+1)\Gamma^2(v/2)} > 0$. This proves part a).

b) Next, using directly Sterling's formula for $\Gamma(v+\alpha)$, $\Gamma(\alpha)$ and $\Gamma(v/2+\alpha)$, one finds that $\lim_{\alpha\to\infty} f(\alpha) = \frac{\Gamma(v+1)}{\Gamma^2(v/2+1)}$, which proves part b).

c) Finally, taking the logarithmic derivative of $f(\alpha)$ yields $\frac{df}{d\alpha} = f(\alpha)\left(\psi(v+\alpha) + \psi(\alpha) - 2\psi(v/2+\alpha)\right)$. This is negative, since $f(\alpha) > 0$ and $\psi(v/2+\alpha) > \frac{1}{2}(\psi(v+\alpha) + \psi(\alpha))$ (because ψ is a convex function). It follows that $f(\alpha) > \lim_{\alpha\to\infty} f(\alpha) = \frac{\Gamma(v+1)}{\Gamma^2(v/2+1)}$. We claim that $g(v) = \frac{\Gamma(v+1)}{\Gamma^2(v/2+1)} > 1$, for any $v > 0$. In fact, the function $g(v)$ is increasing (its derivative is equal to $g(v)(\psi(v+1) - \psi(v/2+1)))$ and $g(0) = 1$. Therefore, $f(\alpha) > 1$, and it follows that $\left(f(\alpha)-1\right)^{-1/2}$ is an increasing function. This completes the proof of part c). ■

Proof of Theorem 23 This is an immediate consequence of Theorem 28. ■

Proof of Theorem 24 Starting with Eq. (10.44), we compute

$$
\begin{aligned}
M_{\mathrm{HK}}^{(v)}(\gamma,\alpha) &= \frac{\Gamma(v/2+1)}{(\gamma+\alpha)^{v/2}} \int_0^\infty w^{v/2}\, {}_1F_1(-v/2, 1, -\frac{\gamma}{w}) \mathcal{G}(w \mid \alpha, 1)\, dw \\
&= \frac{\Gamma(v/2+1)}{(\gamma+\alpha)^{v/2}} \int_0^\infty w^{v/2} e^{-\gamma/w}\, {}_1F_1(1+v/2, 1, \frac{\gamma}{w}) \mathcal{G}(w \mid \alpha, 1)\, dw \\
&= \frac{1}{(\gamma+\alpha)^{v/2}} \frac{\Gamma(v/2+1)}{\Gamma(\alpha)} \sum_{n=0}^{\infty} \frac{(1+v/2)_n}{n!n!} \gamma^n \int_0^\infty w^{v/2+\alpha-n-1} e^{-\gamma/w} e^{-w}\, dw.
\end{aligned}
\tag{10.86}
$$

Using (Erdélyi 1954, I, p. 146, (29)), this is equal to

$$
\begin{aligned}
&\frac{1}{(\gamma+\alpha)^{v/2}} \frac{\Gamma(v/2+1)}{\Gamma(\alpha)} \sum_{n=0}^{\infty} \frac{(1+v/2)_n}{n!n!} \gamma^n 2(\sqrt{\gamma})^{v/2+\alpha-n} K_{v/2+\alpha-n}(2\sqrt{\gamma}) \\
&= \frac{2}{(\gamma+\alpha)^{v/2}} \frac{\Gamma(v/2+1)}{\Gamma(\alpha)} \sum_{n=0}^{\infty} \frac{(1+v/2)_n}{n!n!} (\sqrt{\gamma})^{v/2+\alpha+n} K_{v/2+\alpha-n}(2\sqrt{\gamma}).
\end{aligned}
\tag{10.87}
$$

This completes the proof of Theorem 25. ■

Proof of Theorem 26 a) This follows from Theorem 24, part b), and Theorem 13.

b) From Theorem 24, part a), we know that $M^{(\nu)}_{\mathrm{HK}}(\gamma,\alpha)$ is equal to $\frac{\Gamma(\nu/2+1)}{(\gamma+\alpha)^{\nu/2}}\int_0^\infty w^{\nu/2}\,{}_1F_1(-\nu/2,1,-\gamma/w)\mathcal{G}(w\mid\alpha,1)\,dw$. From Luke (1962, pp. 7–8), we have $\Gamma(\nu/2+1)\,{}_1F_1(-\nu/2,1,-z)=\Gamma(\nu/2+1)e^{-z}\,{}_1F_1(1+\nu/2,1,z)=z^{\nu/2}(1+O(1/z))$, for large values of z. Let $\eta>0$ be a real number (arbitrarily small). Take z_0 sufficiently large so that $(1-\eta)z^{\nu/2}\le\Gamma(\nu/2+1)\,{}_1F_1(-\nu/2,1,-z)\le(1+\eta)z^{\nu/2}$, for any $z\ge z_0$. Then, if $\gamma/w\ge z_0$, i.e. $w\le\gamma/z_0$, we have $(1-\eta)\gamma^{\nu/2}\le\Gamma(\nu/2+1)w^{\nu/2}\,{}_1F_1(-\nu/2,1,-\gamma/w)\le(1+\eta)\gamma^{\nu/2}$. Therefore, the integral $\frac{\Gamma(\nu/2+1)}{(\gamma+\alpha)^{\nu/2}}\int_0^{\gamma/z_0} w^{\nu/2}\,{}_1F_1(-\nu/2,1,-\gamma/w)\mathcal{G}(w\mid\alpha,1)\,dw$ has lower bound $(1-\eta)\frac{\gamma^{\nu/2}}{(\gamma+\alpha)^{\nu/2}}Pr(w\le\gamma/z_0)$ and upper bound $(1+\eta)\frac{\gamma^{\nu/2}}{(\gamma+\alpha)^{\nu/2}}Pr(w\le\gamma/z_0)$. On the other hand, the function ${}_1F_1(-\nu/2,1,-z)$ equals 1 at $z=0$, and hence there is a real number $C>0$ such that $0<{}_1F_1(-\nu/2,1,-\gamma/w)\le C$ for any $w>\gamma/z_0$. Thus, the integral $\frac{\Gamma(\nu/2+1)}{(\gamma+\alpha)^{\nu/2}}\int_{\gamma/z_0}^\infty w^{\nu/2}\,{}_1F_1(-\nu/2,1,-\gamma/w)\mathcal{G}(w\mid\alpha,1)\,dw$ has lower bound 0 and upper bound $\frac{\Gamma(\nu/2+1)}{(\gamma+\alpha)^{\nu/2}}C\frac{\Gamma(\nu/2+\alpha)}{\Gamma(\alpha)}$. But now, $\lim_{\gamma\to\infty}\frac{\gamma^{\nu/2}}{(\gamma+\alpha)^{\nu/2}}=1$, $\lim_{\gamma\to\infty}Pr(w\le\gamma/z_0)=1$, and $\lim_{\gamma\to\infty}\frac{1}{(\gamma+\alpha)^{\nu/2}}=0$. Therefore, we obtain $\liminf_{\gamma\to\infty}M^{(\nu)}_{\mathrm{HK}}(\gamma,\alpha)\ge 1-\eta$ and $\limsup_{\gamma\to\infty}M^{(\nu)}_{\mathrm{HK}}(\gamma,\alpha)\le 1+\eta$. Since η is arbitrarily small, we conclude that $\lim_{\gamma\to\infty}M^{(\nu)}_{\mathrm{HK}}(\gamma,\alpha)=1$.

c) We consider the function $f(\gamma,w)=\frac{{}_1F_1(1+\nu/2,1,\gamma/w)}{e^{\gamma/w}(\gamma+\alpha)^{\nu/2}}$. From Theorem 24, part a), we have $M^{(\nu)}_{\mathrm{HK}}(\gamma,\alpha)=\Gamma(\nu/2+1)\int_0^\infty w^{\nu/2}f(\gamma,w)\mathcal{G}(w\mid\alpha,1)\,dw$. Thus, we obtain $\frac{\partial}{\partial\gamma}M^{(\nu)}_{\mathrm{HK}}(\gamma,\alpha)=\Gamma(\nu/2+1)\int_0^\infty w^{\nu/2}\frac{\partial}{\partial\gamma}f(\gamma,w)\mathcal{G}(w\mid\alpha,1)\,dw$.

We compute the value of $\frac{\partial}{\partial\gamma}f(\gamma,w)$ as

$$\frac{\frac{d}{dz}{}_1F_1(1+\nu/2,1,\gamma/w)w^{-1}-{}_1F_1(1+\nu/2,1,\gamma/w)\left(w^{-1}+\frac{\nu}{2}(\gamma+\alpha)^{-1}\right)}{e^{\gamma/w}(\gamma+\alpha)^{\nu/2}}. \tag{10.88}$$

Using (Gradshteyn and Ryshik 1994, 9.213, p. 1086) and (Gradshteyn and Ryshik 1994, 9.212(3), p. 1086), we have $\frac{d}{dz}{}_1F_1(1+\nu/2,1,\gamma/w)=(1+\nu/2)\,{}_1F_1(2+\nu/2,2,\gamma/w)=\frac{\nu}{2}{}_1F_1(1+\nu/2,2,\gamma/w)+{}_1F_1(1+\nu/2,1,\gamma/w)$. So, we obtain after algebraic simplifications

$$\frac{\nu/2}{e^{\gamma/w}(\gamma+\alpha)^{\nu/2+1}}\left\{{}_1F_1(1+\nu/2,2,\gamma/w)\left(\frac{\gamma}{w}+\frac{\alpha}{w}\right)-{}_1F_1(1+\nu/2,1,\gamma/w)\right\}. \tag{10.89}$$

Using (Gradshteyn and Ryshik 1994, 9.212(2), p. 1086), we have $\frac{\gamma}{w}{}_1F_1(1+v/2,2,\gamma/w) - {}_1F_1(1+v/2,1,\gamma/w) = -\,{}_1F_1(v/2,1,\gamma/w)$. Therefore, we finally obtain

$$\frac{\alpha v/2}{e^{\gamma/w}(\gamma+\alpha)^{v/2+1}w}\left\{{}_1F_1(1+v/2,2,\gamma/w) - \frac{w}{\alpha}{}_1F_1(v/2,1,\gamma/w)\right\}. \tag{10.90}$$

Now, let $v/2 < 1$. Then, we obtain the strict lower bound for $\frac{\partial}{\partial\gamma}f(\gamma,w)$

$$\frac{\alpha v/2}{e^{\gamma/w}(\gamma+\alpha)^{v/2+1}w}\left\{{}_1F_1(1+v/2,2,\gamma/w) - \frac{w}{\alpha}{}_1F_1(1+v/2,2,\gamma/w)\right\}. \tag{10.91}$$

Consider the function $g(\gamma,w) = \Gamma(v/2+1)w^{v/2}\frac{\alpha v/2}{e^{\gamma/w}(\gamma+\alpha)^{v/2+1}w}{}_1F_1(1+v/2,2,\gamma/w)$. We have shown that

$$\frac{\partial}{\partial\gamma}M_{\mathrm{HK}}^{(v)}(\gamma,\alpha) > \int_0^\infty g(\gamma,w)\mathcal{G}(w\,|\,\alpha,1)\,dw - \int_0^\infty \frac{w}{\alpha}g(\gamma,w)\mathcal{G}(w\,|\,\alpha,1)\,dw. \tag{10.92}$$

But, $\frac{w}{\alpha}\mathcal{G}(w\,|\,\alpha,1) = \mathcal{G}(w\,|\,\alpha+1,1)$. So, we obtain

$$\frac{\partial}{\partial\gamma}M_{\mathrm{HK}}^{(v)}(\gamma,\alpha) > \int_0^\infty g(\gamma,w)\mathcal{G}(w\,|\,\alpha,1)\,dw - \int_0^\infty g(\gamma,w)\mathcal{G}(w\,|\,\alpha+1,1)\,dw. \tag{10.93}$$

Thus, we want to show that $\int_0^\infty g(\gamma,w)\mathcal{G}(w\,|\,\alpha,1)\,dw - \int_0^\infty g(\gamma,w)\mathcal{G}(w\,|\,\alpha+1,1)\,dw \geq 0$. Ignoring the positive factor $\Gamma(v/2+1)\frac{\alpha v/2}{(\gamma+\alpha)^{v/2+1}}$, we are thus lead to the function $h(\gamma,w) = \frac{w^{v/2-1}\,{}_1F_1(1+v/2,2,\gamma/w)}{e^{\gamma/w}}$, and we show that $\int_0^\infty h(\gamma,w)\mathcal{G}(w\,|\,\alpha,1)\,dw - \int_0^\infty h(\gamma,w)\mathcal{G}(w\,|\,\alpha+1,1)\,dw \geq 0$ as follows. In Lemma 6, we show that $h(\gamma,w)$ is decreasing in the variable w, if $v/2 < 1$. Then, in Lemma 7, we show that for any decreasing positive function $H(w)$, we have $\int_0^\infty H(w)\mathcal{G}(w\,|\,\alpha,1)\,dw - \int_0^\infty H(w)\mathcal{G}(w\,|\,\alpha+1,1)\,dw \geq 0$.

Next, let $v/2 > 1$. Then, we obtain the strict upper bound for $\frac{\partial}{\partial\gamma}f(\gamma,w)$

$$\frac{\alpha v/2}{e^{\gamma/w}(\gamma+\alpha)^{v/2+1}w}\left\{{}_1F_1(1+v/2,2,\gamma/w) - \frac{w}{\alpha}{}_1F_1(1+v/2,2,\gamma/w)\right\}. \tag{10.94}$$

The same argument as above (but with reversed inequalities) leads to

$$\frac{\partial}{\partial\gamma}M_{\mathrm{HK}}^{(v)}(\gamma,\alpha) < \int_0^\infty g(\gamma,w)\mathcal{G}(w\,|\,\alpha,1)\,dw - \int_0^\infty g(\gamma,w)\mathcal{G}(w\,|\,\alpha+1,1)\,dw, \tag{10.95}$$

where the function $g(\gamma, w)$ is defined as above. So, in this case, we want to show that $\int_0^\infty h(\gamma, w)\mathcal{G}(w \mid \alpha, 1)\, dw - \int_0^\infty h(\gamma, w)\mathcal{G}(w \mid \alpha + 1, 1)\, dw \leq 0$, where $h(\gamma, w)$ is defined as above. But, this is implied by Lemmas 6 and 7 (case $v/2 > 1$). This completes the proof of the theorem. ■

Lemma 6*a) If* $v/2 < 1$, *the function* $h(\gamma, w) = \frac{w^{v/2-1}\,{}_1F_1(1+v/2,2,\gamma/w)}{e^{\gamma/w}}$ *is decreasing in the variable* w.

b) If $v/2 > 1$, *the function* $h(\gamma, w)$ *is increasing in the variable* w.

Proof Using the change of variable $x = \gamma/w$, we consider the function $F(x) = \frac{{}_1F_1(1+v/2,2,x)}{e^x x^{v/2-1}}$. So, we want to show that $F(x)$ is increasing if $v/2 < 1$ and $F(x)$ is decreasing if $v/2 > 1$ (the function $x = \gamma/w$ is decreasing in the variable w).

We compute

$$\frac{d}{dx}F(x) = \frac{\frac{d}{dz}{}_1F_1(1 + v/2, 2, x) - {}_1F_1(1 + v/2, 2, x)\left(1 + (\frac{v}{2} - 1)x^{-1}\right)}{e^x x^{v/2-1}}. \tag{10.96}$$

Using (Gradshteyn and Ryshik 1994, 9.213, p. 1086) and (Gradshteyn and Ryshik 1994, 9.212(3), p. 1086), we have $\frac{d}{dz}{}_1F_1(1 + v/2, 2, x) = \frac{(1+v/2)}{2}{}_1F_1(2 + v/2, 3, x) = \frac{(v/2-1)}{2}{}_1F_1(1 + v/2, 3, x) + \frac{2}{2}{}_1F_1(1 + v/2, 2, x)$.

So, we obtain after algebraic simplifications

$$\frac{(v/2 - 1)}{e^x x^{v/2}}\left\{\frac{x}{2}{}_1F_1(1 + v/2, 3, x) - {}_1F_1(1 + v/2, 2, x)\right\}. \tag{10.97}$$

Using (Gradshteyn and Ryshik 1994, 9.212(2), p. 1086), we finally obtain

$$-\frac{(v/2 - 1)}{e^x x^{v/2}}{}_1F_1(1 + v/2, 1, x). \tag{10.98}$$

The result is now clear. ■

Lemma 7 *a) Let* $H(w)$ *be a decreasing positive function. Then, one has* $\int_0^\infty H(w)\mathcal{G}(w \mid \alpha, 1)\, dw - \int_0^\infty H(w)\mathcal{G}(w \mid \alpha + 1, 1)\, dw \geq 0$.

b) Let $H(w)$ *be an increasing positive function. Then, one has* $\int_0^\infty H(w)\mathcal{G}(w \mid \alpha, 1)\, dw - \int_0^\infty H(w)\mathcal{G}(w \mid \alpha + 1, 1)\, dw \leq 0$.

Proof a) Since $H(w)$ is a positive decreasing function, we can approximate it by functions of the form $\sum_{n=1}^N a_n B(b_n, w)$, where $a_n \geq 0$, $b_n > 0$, and $B(b, w)$ is equal to 1, if $w \in [0, b]$, and $B(b, w) = 0$, if $w > b$. Now, $\int_0^\infty B(b, w)\mathcal{G}(w \mid \alpha, 1)\, dw = \int_0^b \mathcal{G}(w \mid \alpha, 1)\, dw = 1 - \frac{\Gamma(\alpha,b)}{\Gamma(\alpha)}$, where $\Gamma(\alpha, b)$ is the incomplete Euler gamma function. But the function $1 - \frac{\Gamma(\alpha,b)}{\Gamma(\alpha)}$ is decreasing. Therefore, $\int_0^\infty B(b, w)\mathcal{G}(w \mid \alpha, 1)\, dw > \int_0^\infty B(b, w)\mathcal{G}(w \mid \alpha + 1, 1)\, dw$, and we are done.

b) Since $H(w)$ is a positive increasing function,, we can approximate $H(w)$ by functions of the form $\sum_{n=1}^{N} a_n\,(1 - B(b_n, w))$, where $a_n \geq 0$, $b_n > 0$. Now, $\int_0^\infty (1 - B(b, w))\mathcal{G}(w \mid \alpha, 1)\,dw = \frac{\Gamma(\alpha,b)}{\Gamma(\alpha)}$, and we are done. ■

Proof of Corollary 2. Let $v/2 < 1$. Since $0 < M < 1$, we conclude from Theorem 26, using the Intermediate Value Theorem, that for any $\alpha > 0$ such that $M_K(\alpha) \leq M$, there is a unique value of $\gamma \geq 0$ for which $M_{HK}^{(v)}(\gamma, \alpha) = M$. Thus, if $M \geq \Gamma(v/2 + 1)$, α has no restrictions, because $M_K^{(v)}(\alpha) < \Gamma(v/2 + 1)$ for any $\alpha > 0$ (Theorem 21). On the other hand, if $M_K(\alpha) < M$, let α_0 be the unique solution to the equation $M_K^{(v)}(\alpha_0) = M$ (Theorem 21). Then, using once more Theorem 21, we obtain that $M_K^{(v)}(\alpha) < M$ if and only if $\alpha \leq \alpha_0$. Henceforth, if $M_K^{(v)}(\alpha) < M$, the domain of the function $\gamma_M^{(v)}(\alpha)$ is the interval $(0, \alpha_0]$

The case $v/2 > 1$ is handled similarly, but with reversed inequalities. ■

Proof of Lemma 4. Part a). From the definition in Eq. (10.6), the distribution $P_{\mathrm{HK}}(A \mid \varepsilon, \sigma^2, \alpha)$ is equal to $\int_0^\infty P_{\mathrm{Ri}}(A \mid \varepsilon, \sigma^2 w)\mathcal{G}(w \mid \alpha, 1)d\,w$. Using the identity $I_0(z) = \frac{1}{\pi}\int_0^\pi e^{z\cos\theta} d\,\theta$ from Abramowitz and Stegun (1972, Eq. (9.6.16), p. 376) and the definition of the Rice distribution (10.2) , we can express $P_{\mathrm{Ri}}(A \mid \varepsilon, \sigma^2 w)$ in the form $\frac{1}{\pi}\int_0^\pi \frac{A}{\sigma^2 w}\exp\left(\frac{\varepsilon}{\sigma^2 w} A\cos\theta\right)\exp\left(-\frac{(\varepsilon^2 + A^2)}{2\sigma^2 w}\right)d\,\theta$. It follows that $P_{\mathrm{HK}}(A \mid \varepsilon, \sigma^2, \alpha)$ can be written as

$$\frac{1}{\pi}\int_0^\pi \left\{\int_0^\infty \frac{A}{\sigma^2 w}\exp\left(\frac{\varepsilon}{\sigma^2 w}A\cos\theta\right)\exp\left(-\frac{(\varepsilon^2 + A^2)}{2\sigma^2 w}\right)\mathcal{G}(w \mid \alpha, 1)d\,w\right\}d\;\theta, \tag{10.99}$$

which yields Eq. (10.56) after evaluation of the inner integral.

Part b). Using Eq. (10.56), the partial derivative of the homodyned K-distribution with respect to ε is equal to

$$\begin{aligned}
&\frac{1}{\pi}\int_0^\pi \frac{2A}{\sigma^2\Gamma(\alpha)}\frac{\partial}{\partial\varepsilon}\left\{\left(\frac{X(\theta)}{2}\right)^{\alpha-1} K_{\alpha-1}\big(X(\theta)\big)\right\}d\;\theta \\
&= \frac{1}{\pi}\int_0^\pi \frac{2A}{\sigma^2\Gamma(\alpha)}\left\{\frac{(\alpha-1)}{2}\left(\frac{X(\theta)}{2}\right)^{\alpha-2} K_{\alpha-1}\big(X(\theta)\big)\right. \\
&\qquad \left. + \left(\frac{X(\theta)}{2}\right)^{\alpha-1}\frac{d}{d\,z}K_{\alpha-1}\big(X(\theta)\big)\right\}\frac{\partial}{\partial\varepsilon}X(\theta)d\;\theta \\
&= -\frac{1}{\pi}\int_0^\pi \frac{2A}{\sigma^2\Gamma(\alpha)}\left(\frac{X(\theta)}{2}\right)^{\alpha-1} K_{\alpha-2}\big(X(\theta)\big)2\,\frac{(\varepsilon - A\cos\theta)}{\sigma^2 X(\theta)}d\;\theta.
\end{aligned} \tag{10.100}$$

Here, we have used the identity $z\frac{d}{d\,z}K_{\alpha-1}(z) + (\alpha-1)K_{\alpha-1}(z) = -z\,K_{\alpha-2}(z)$ (Abramowitz and Stegun 1972, Eq. (9.6.26), 2nd identity, p. 376) and algebraic simplifications.

Part c). Using Eq. (10.56), the partial derivative of the homodyned K-distribution with respect to σ^2 is equal to

$$\begin{aligned}
&\frac{1}{\pi}\int_0^{\pi} \frac{2A}{\Gamma(\alpha)} \frac{\partial}{\partial\sigma^2}\left\{\frac{1}{\sigma^2}\left(\frac{X(\theta)}{2}\right)^{\alpha-1} K_{\alpha-1}\big(X(\theta)\big)\right\} d\,\theta \\
&= -\frac{1}{\sigma^2}\frac{1}{\pi}\int_0^{\pi} \frac{2A}{\sigma^2\Gamma(\alpha)}\left(\frac{X(\theta)}{2}\right)^{\alpha-1} K_{\alpha-1}\big(X(\theta)\big) d\,\theta \\
&\quad + \frac{1}{\pi}\int_0^{\pi} \frac{2A}{\sigma^2\Gamma(\alpha)}\left\{\frac{(\alpha-1)}{2}\left(\frac{X(\theta)}{2}\right)^{\alpha-2} K_{\alpha-1}\big(X(\theta)\big)\right. \\
&\quad \left. + \left(\frac{X(\theta)}{2}\right)^{\alpha-1} \frac{d}{d\,z} K_{\alpha-1}\big(X(\theta)\big)\right\} \frac{\partial}{\partial\sigma^2} X(\theta) d\,\theta \\
&= -\frac{1}{\sigma^2}\frac{1}{\pi}\int_0^{\pi} \frac{2A}{\sigma^2\Gamma(\alpha)}\left(\frac{X(\theta)}{2}\right)^{\alpha-1} K_{\alpha-1}\big(X(\theta)\big) d\,\theta \\
&\quad - \frac{1}{\sigma^2}\frac{1}{\pi}\int_0^{\pi} \frac{2A}{\sigma^2\Gamma(\alpha)}\left\{(\alpha-1)\left(\frac{X(\theta)}{2}\right)^{\alpha-2} K_{\alpha-1}\big(X(\theta)\big)\right. \\
&\quad \left. - \left(\frac{X(\theta)}{2}\right)^{\alpha-1} K_{\alpha}\big(X(\theta)\big)\right\}\left(\frac{X(\theta)}{2}\right) d\,\theta \\
&= -\frac{\alpha}{\sigma^2}\frac{1}{\pi}\int_0^{\pi} \frac{2A}{\sigma^2\Gamma(\alpha)}\left(\frac{X(\theta)}{2}\right)^{\alpha-1} K_{\alpha-1}\big(X(\theta)\big) d\,\theta \\
&\quad + \frac{1}{\sigma^2}\frac{1}{\pi}\int_0^{\pi} \frac{2A}{\sigma^2\Gamma(\alpha)}\left(\frac{X(\theta)}{2}\right)^{\alpha} K_{\alpha}\big(X(\theta)\big) d\,\theta.
\end{aligned} \tag{10.101}$$

Here, we have used the identity $\frac{z}{2}\frac{d}{dz}K_{\alpha-1}(z) = -\frac{z}{2}K_{\alpha}(z) + \frac{(\alpha-1)}{2}K_{\alpha-1}(z)$ (Abramowitz and Stegun 1972, Eq. (9.6.26), 4th identity, p. 376) and algebraic simplifications.

Part d). Eq. (10.59) follows from part a) upon taking the logarithmic derivative of the integrand in Eq. (10.56). ■

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